18.06 (Spring 14) Problem Set 5

This problem set is due Thursday, March 20, 2014 by 4pm in E17-131. The problems are out of the 4th edition of the textbook. This homework has 8 questions worth 100 points in total. Please WRITE NEATLY. You may discuss with others (and your TA), but you must turn in your own writing.

1. Least squares approximation: if \( b = (5, 13, 17) \) at \( t = (-1, 1, 2) \) and the best line \( C + Dt \), and the vectors \( p \) and \( e \). Check that \( e^T p = 0 \).

**Solution:**

\[
A = \begin{bmatrix}
1 & -1 \\
1 & 1 \\
1 & 2 \\
\end{bmatrix}, \quad x = \begin{bmatrix} C \\ D \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix}
\]

The system is solvable and has solution \( x = \begin{bmatrix} 9 \\ 4 \end{bmatrix} \). Hence the best line is \( 9 + 4t \), which fits perfectly giving zero error \( e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \), and \( p = b \).

2. Least squares approximation: If \( b \) takes values 0, 1, 3, respectively 4 at the corners \( (x, y) = \{(1, 0), (0, 1), (-1, 0), (0, -1)\} \) of a square, find the best least squares fit to \( b \) by a plane \( C + Dx + Ey \).

**Solution:**

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & -1 & 0 \\
1 & 0 & -1 \\
\end{bmatrix}, \quad x = \begin{bmatrix} C \\ D \\ E \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}
\]

The system is not solvable. We solve \( A^T A \tilde{x} = A^T b \), where

\[
A^T A = \begin{bmatrix}
4 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}, \quad A^T b = \begin{bmatrix} 8 \\ -3 \\ -3 \end{bmatrix}
\]

which gives

\[
\tilde{x} = \begin{bmatrix} 2 \\ -3/2 \\ -3/2 \end{bmatrix}
\]

So the best fitting plane is \( 2 - 3/2x - 3/2y \). (You can also check by running \textit{lsqilin}(A, b) in MATLAB.)

3. Consider the reflection matrix \( R = I - 2P \), where \( P \) is a projection.
(a) Show that $R^2 = I$.
(b) For vectors $v$ in $C(P)$, what is $Rv$?
(c) For vectors $w$ orthogonal to $C(P)$, what is $Rw$?

**Solution:**

(a) $R^2 = (I - 2P)^2 = I^2 - 2PI - 2IP + 4P^2 = I - 4P + 4P = I$. (We used the fact that $P^2 = P$, since $P$ is a projection).

(b) Let $v = Px \in C(P)$. Then $Rv = (I - 2P)v = v - 2Pv = v - 2P(Px) = v - 2P^2x = v - 2Px = v - 2v = -v$.

(c) Let $w = (I - P)x \in C(P)^\perp$. Then $Rw = (I - 2P)w = w - 2Pw = w - 2P(I - P)x = w - 2Px + 2P^2x = w - 2Px + 2Px = w$.

4. If $q_1, q_2, \ldots, q_n$ are orthonormal vectors in $\mathbb{R}^n$, what combination of the $q$’s produces a given vector $v$? How do you know that the $q$’s are a basis for $\mathbb{R}^n$?

**Solution:**

$$v = \sum_{i=1}^{n} (q_i^Tv) \cdot q_i$$

The vectors are orthonormal, hence linearly independent, so they form a basis. Another way to verify is to look at the projection matrix on the span of the $q_i$’s. Consider the matrix with the $q_i$’s as column vectors $A = [q_1 \ldots q_n]$. Then $A^TA = I$. Hence the projection matrix onto their span is $P = A(A^TA)^{-1}A^T = AA^T = (A^TA)^T = I$.

5. (a) In class, you may have seen that in the projection matrix formula $P = A(A^TA)^{-1}A^T$, we always want $A$ to have linearly independent columns. Why?
(b) Given an $m \times n$ matrix $A$ ($m > n$), what is the projection matrix $P_{null}$ onto the nullspace of $A^T$?

**Solution:**

(a) If the columns are not linearly independent, $A$ does not have full column rank. So $A^TA$ is not full rank, and hence not invertible.

(b) Recall that $N(A^T) \perp C(A)$. Let $P$ be the projection on $C(A)$. Hence $P = A(A^TA)^{-1}A^T$. The projection onto the null space of $A^T$ is perpendicular to that on the column space of $A$, hence $P_{null} = I - P = I - A(A^TA)^{-1}A^T$. 

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