18.06 (Spring 14) Problem Set 6

This problem set is due Thursday, April 3, 2014 by 4pm in E17-131. The problems are out of the 4th edition of the textbook. This homework has 8 questions worth 80 points in total. Please WRITE NEATLY. You may discuss with others (and your TA), but you must turn in your own writing.

1. Suppose every vector in \( \mathbb{R}^2 \) is multiplied by the matrix \( A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \).

   (a) Describe the shape that comes from multiplying every vector in the square of side 2 centered at \((0, 0)\) by \( A \).

   (b) What is the area of that shape?

Solution:

(a) We check where the corners of the squares are being sent by \( A \):

\[
A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, A \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}, A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}
\]

Every point in the interior of the square is a convex combination of the corners, so (since we only apply a linear transformation) all the points in the interior of the square will be sent to convex combinations of the vectors \( \left\{ \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\} \).

So the shape we obtain is the parallelogram with the corners given above.

(b) \( \det(A) \cdot \text{area of the square} = 10 \cdot 4 = 40 \)

2. Suppose you have any shape with area 1 in the \( xy \) plane. If you multiply every vector in that shape by the same \( A \) as above, then the area is multiplied by the determinant of \( A \). Why is this?

Solution:

The area of the initial shape \( S \) in the plane \( xy \) is \( \int \int_{S(xy)} dA \). The area of the shape after applying the transformation is \( \int \int_{AS(xy)} dA \). After a quick glance we see that the Jacobian matrix \( J = A \). Hence after a change of variable \( \int \int_{AS(xy)} dA = \int \int_{S(xy)} \det(J) dA = \det(A) \cdot \int \int_{S(xy)} dA \).

A simpler, yet not rigorous solution is to argue that the statement holds for a parallelogram, then extend it to arbitrary shapes by partitioning them into infinitesimally small parallelograms. Indeed, given a parallelogram determined by vectors \( u, v \), its area is \( \det([u \ v]) \). Then, after applying \( A \), the area becomes \( \det([Au \ Av]) = \det(A[u \ v]) = \det(A) \det([u \ v]) \), so the area gets multiplied by \( A \).

3. Find the determinant of the \( 3 \times 3 \) matrix \( A \) whose \((i, j)\) entry is \( 3^{i-j} \). Find the inverse of \( A \) using the cofactor matrix and the determinant \((A^{-1} = C^T / \det A)\).
Solution:

\[ A = \begin{bmatrix} 1 & 3 & 9 \\ 3 & 1 & 3 \\ 9 & 3 & 1 \end{bmatrix}, \quad \det(A) = 1^3 + 3^2 \cdot 9 + 3^2 \cdot 9 - 9^2 - 3^2 - 3^2 = 64 \]

\[ C = \begin{bmatrix} -8 & 24 & 0 \\ 24 & -80 & 24 \\ 0 & 24 & -8 \end{bmatrix} \]

\[ A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{8} \begin{bmatrix} -1 & 3 & 0 \\ 3 & -10 & 3 \\ 0 & 3 & -1 \end{bmatrix} \]

4. (p. 252, problem 11) Suppose that \( CD = -DC \) and find the flaw in this reasoning:
Taking determinants gives \( |C| \cdot |D| = -|D| \cdot |C| \). Therefore \( |C| = 0 \) or \( |D| = 0 \). One or both of the matrices must be singular.

Solution:

\( CD = -DC \Rightarrow \det(CD) = (-1)^n \det(DC) \), and not \( -\det(DC) \). If \( n \) is even, \( CD \) can be invertible.

5. Prove that every orthogonal matrix \( Q \) has determinant +1 or –1.

Solution:

\( Q \) is orthogonal, hence \( Q^T Q = I \). Hence \( \det(Q)^2 = \det(Q^T) \det(Q) = \det(Q^T Q) = \det(I) = 1 \). So \( \det(Q) \in \{ \pm 1 \} \).

6. Find the determinant of \( I + M \), if \( M \) is the rank one matrix \( M = vv^T \), where \( v \) is a column vector \( v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \).

Solution:

\[ I + M = \begin{bmatrix} 1 + a^2 & ab & ac \\ ab & 1 + b^2 & bc \\ ac & bc & 1 + c^2 \end{bmatrix} \]

\[ \det(I + M) = (1 + a^2) \begin{vmatrix} 1 + b^2 & bc \\ bc & 1 + c^2 \end{vmatrix} - ab \begin{vmatrix} ab & bc \\ ac & 1 + c^2 \end{vmatrix} + ac \begin{vmatrix} ab & 1 + b^2 \\ ac & bc \end{vmatrix} \]

\[ = (1 + a^2)(1 + b^2 + c^2) - ab \cdot ab + ac \cdot (-ac) \]

\[ = 1 + a^2 + b^2 + c^2 \]
Side note: If you know Sylvester’s determinant theorem, the problem becomes incredibly easy. Applied to this particular instance, we get that \( \det(I + vv^T) = \det(1 + v^Tv) = 1 + \|v\|^2 \).

7. Suppose \( A_n \) is the \( n \times n \) symmetric tridiagonal matrix with a subdiagonal of 1’s, a main diagonal of 3’s, and a superdiagonal of 1’s. By cofactors of row 1, connect the determinant of \( A_n \) to the determinants of \( A_{n-1} \) and \( A_{n-2} \).

Solution:

The first row has nonzero elements only on the first two columns. The first cofactor is \( C_{11} = A_{n-1} \), the second is \( C_{12} = \begin{bmatrix} 1 & 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ A_{n-2} & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \). We can compute \( \det(C_{12}) = 1 \cdot \det(A_{n-2}) + 1 \cdot 0 = \det(A_{n-2}) \) (since when taking the second cofactor we get only zeroes in the first column). Putting everything together, we finally get \( \det(A_n) = 3 \cdot \det(A_{n-1}) - 1 \cdot \det(A_{n-2}) \).

8. (a) Find orthonormal vectors \( q_1, q_2, q_3 \) such that \( q_1 \) and \( q_2 \) span the column space of

\[
A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & -2 \end{bmatrix}.
\]

Use Gram-Schmidt.

(b) Which of the four fundamental subspaces for \( A \) will contain \( q_3 \)?

(c) Solve \( Ax = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \) by least squares.

Solution:

(a) Let \( c_1 \) and \( c_2 \) be the columns of \( A \). Then \( q_1 = \frac{c_1}{\|c_1\|} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, q_2 = \frac{c_2 - q_1 \cdot (c_2^T q_1)}{\|c_2 - q_1 \cdot (c_2^T q_1)\|} = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix} \). To get \( q_3 \), pick an arbitrary vector \( c_3 \) that is obviously not in the column space of \( A \), and continue doing Gram-Schmidt. For example we can pick \( c_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \), then compute \( q_3 = \frac{c_3 - q_1 \cdot (c_3^T q_1) - q_2 \cdot (c_2^T q_1)}{\|c_3 - q_1 \cdot (c_3^T q_1) - q_2 \cdot (c_2^T q_1)\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \).
(b) $q_3 \in N(A^T)$, since $N(A^T) \perp C(A)$, and $q_3 \perp C(A)$.

(c) Solve $A^T Ax = A^T \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$, equivalent to $\begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix} x = \begin{bmatrix} 14 \\ 6 \end{bmatrix}$. Hence $x = \begin{bmatrix} 9 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 14 \\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ -1 \\ -1 \\ 9 \end{bmatrix} \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}.$

9. MATLAB problems: Please go to lms.mitx.mit.edu to finish this part.