Problem Set 8, 18.06

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Problem 1 (6.4 §5). Find an orthogonal matrix $Q$ that diagonalizes the symmetric matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}.$$

Solution: The characteristic polynomial of the matrix is $\lambda(\lambda - 1)(\lambda + 1)$, so the eigenvalues are 0, $-3$ and 3. Their respective normalized eigenvectors are given in order as the columns of $Q$:

$$Q = \frac{1}{3} \begin{pmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{pmatrix}.$$

Problem 2 (6.4 §10). Here is a “quick” proof that the eigenvalues of all real matrices are real:

False proof $Ax = \lambda x$ gives $x^T Ax = \lambda x^T x$ so $\lambda = \frac{x^T Ax}{x^T x}$ is real.

Find the flaw in this reasoning—a hidden assumption that is not justified. You could test those steps on the $90^\circ$ rotation matrix $[0 -1; 1 0]$ with $\lambda = i$ and $x = (i,1)$.

Solution: The problem is that $\frac{x^T Ax}{x^T x}$ is not necessarily real when $x$ is not real, and eigenvectors are not necessarily real.

Problem 3 (6.4 §23). Which of these classes of matrixes do $A$ and $B$ belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad B = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Which of these factorizations are possible for $A$ and $B$: $LU$, $QR$, $S\Lambda S^{-1}$, $Q\Lambda Q^{-1}$?

Solution: $A$ is permutation, so it is also invertible, orthogonal and Markov. It is symmetric, so it is also diagonalizable. However, $A^2 = I \neq A$ so $A$ is not projection. $A$ does not allow $LU$ as we need to permute its rows to make it $LU$-factorable, but by the Spectral Theorem it allows $Q\Lambda Q^{-1}$ and then $S\Lambda S^{-1}$. It clearly allows $QR$ with $R$ having strictly positive diagonal entries since it is invertible, in fact, $A = QI$. 

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B is Markov, and idempotent (i.e. $B^2 = B$), so it is projection, and therefore diagonalizable. It is clearly neither invertible, nor orthogonal, nor permutation. Again, by the Spectral Theorem, $B$ allows $Q\Lambda Q^{-1}$ and so $S\Lambda S^{-1}$. The columns of $B$ are not independent, so $B$ does not allow the $QR$ factorization. With regards to $LU$, we could write down:

$$B = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Problem 4 (6.5 §7).** Test to see if $R^T R$ is positive definite in each case:

$R = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 \end{pmatrix}$.

**Solution:** Clearly these products are all semidefinite-positive, so the question translates to asking for which cases is the matrix $R^T R$ invertible. It is then easy to check that the first two $R$’s satisfy the condition, but not the third.

**Problem 5 (6.5 §12).** For what numbers $c$ and $d$ are $A$ and $B$ positive definite? Test the 3 determinants:

$A = \begin{pmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{pmatrix}$.

**Solution:** For $A$, the upper central minors are $c$, $c^2 - 1$ and $c^3 + 2 - 3c = (c-1)(c^2 + c - 2)$. These are all positive if and only if $c > 1$. For $B$, these minors are $1$, $d - 4$ and $4(3 - d)$. No real values of $d$ can simultaneously make these three numbers positive.

**Problem 6 (6.5 §14).** If $A$ is positive definite then $A^{-1}$ is positive definite.

Best proof: The eigenvalues of $A^{-1}$ are positive because they are the multiplicative inverses of the eigenvalues of $A$, which are positive.

Second proof (only for 2 by 2): The entries of $A^{-1} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$ pass the determinant tests: $ac - b^2 > 0$ because this is the determinant of $A$. Moreover, as $a > 0$ by the tests for $A$, then $c > 0$ as well for otherwise $ac - b^2 \leq 0$. Then, the central upper minors of $A^{-1}$ are $c > 0$ and $ac - b^2 > 0$.

**Problem 7 (6.5 §19).** Suppose that all the eigenvalues $\lambda$ of a diagonalizable symmetric matrix $A$ satisfy that $\lambda > 0$. Show that then $x^T Ax > 0$ for every nonzero vector $x$. Note that $x$ is not necessarily an eigenvector of $A$, so write $x$ as a linear combination of these eigenvectors and explain why all the “cross terms” are $x_i^T x_j = 0$. Then, argue that $x^T Ax$ is

$$(c_1x + \cdots + c_nx_n)(c_1\lambda_1x_1 + \cdots + c_n\lambda_nx_n) = c_1^2\lambda_1^2x_1^2 + \cdots + c_n^2\lambda_n^2x_n^2 > 0.$$

**Solution:** As $A$ is diagonalizable, any non-zero $x$ can be written as a linear combination of the eigenvectors of $x_1, \ldots, x_n$. We can choose an orthogonal basis for each separate eigenspace of $A$ corresponding to one eigenvalue by using
Gram-Schmidt, and moreover, the eigenspaces corresponding to different eigenvalues are orthogonal because $A$ is symmetric. Then, we can choose $x_1, \ldots, x_n$ so that $x_i^T x_j = 0$ for all $i \neq j$. The final verification is easy since every $\lambda_i > 0$ and every $c_i^2 \geq 0$, and at least one $c_i^2 > 0$ because $x$ is non-zero.

**Problem 8 (6.5 28).** Without multiplying

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},$$

find:

a) the determinant of $A$,

b) the eigenvalues of $A$,

c) the eigenvectors of $A$,

d) the reason why $A$ is symmetric positive definite.

**Solution:** What we have to notice is that we are given a $QΛQ^{-1} = QΛQ^T$ decomposition of the matrix, so all the information that we want can be read off from it. For a) the determinant is equal the determinant of the diagonal matrix $2 \cdot 5 = 10$. For b), we have that the eigenvalues are 2 and 5. For c), the eigenvectors are the columns of $Q$, so $[\cos \theta \ \sin \theta]$' and $[-\sin \theta \ \cos \theta]'$. The matrix is clearly symmetric since $(QΛQ^T)^T = QΛQ^T$ and its eigenvalues are positive, so it is positive-definite.

**Solution for Matlab problem:** Given that the derivative of the determinant is equal to a non-zero scalar times the cofactor of either one of the $x$'s, then maximizing the determinant makes these cofactors equal to 0. Then, by Cramer’s Rule, which expresses the inverse of an invertible matrix in terms of its cofactors, and noticing that $B_{31}^{-1}$ and $B_{13}^{-1}$ are then both equal to these cofactors divided by $\det(A)$, we see that $B_{31}^{-1} = B_{13}^{-1} = 0$ because the cofactors are zero when $x$ is optimal.