18.06 Exam 1 review

1. Fill the blanks: for an \( n \times n \) invertible matrix \( A \), the column space \( C(A) = \ldots \), the null space \( N(A) = \ldots \), the pivot columns are \( \ldots \), \( R = rref(A) \) equals \( \ldots \), and the solution to \( Ax = b \) is \( \ldots \).

2. Answer to the same question as in Problem 1, when \( A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 \end{bmatrix} \). What are the special solutions to \( Ax = 0 \)?

3. Answer to the same question as in Problem 1, when \( A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \). What are the special solutions to \( Ax = 0 \)?

4. If \( E \) is a square and invertible matrix,
   (a) How is \( C(EA) \) related to \( C(A) \)?
   (b) How is \( N(EA) \) related to \( N(A) \)?
   (c) How is \( rref(EA) \) related to \( rref(A) \)?

5. If \( A \) is a \( 5 \times 6 \) matrix and \( R = rref(A) \),
   (a) Why are there nonzero solutions to \( Ax = 0 \)?
   (b) How is \( C([A \ A]) \) related to \( C(A) \)? (Note that \( [A \ A] \) is a \( 5 \times 12 \) matrix.)
   (c) There are at least \( \ldots \) special solutions to \( [A \ A] \begin{bmatrix} x \\ y \end{bmatrix} = [0] \).

6. If \( P \) is a permutation matrix, explain why \( P^N = I \) holds for some positive integer \( N \).

7. (a) If \( A \) and \( B \) are \( 4 \times 4 \) and \( AB \) is invertible, show that \( A \) is invertible.
   (b) A \( 5 \times 4 \) matrix times a \( 4 \times 5 \) matrix cannot produce an invertible \( 5 \times 5 \) matrix. Why not?

8. Here are 8 equivalent statements (plus 2 more that involve \( A^T A \) — coming soon).
   (1) The columns of \( A \) are independent
   (2) The rows of \( A \) span \( \mathbb{R}^n \)
   (3) The rank of \( A \) is \( n \): “full column rank”
   (4) All the columns of \( A \) are pivot columns so \( R = \begin{bmatrix} I \\ 0 \end{bmatrix} \)
   (5) The nullspace \( N(A) \) contains only the zero vector
   (6) The row space \( C(A^T) \) is all of \( \mathbb{R}^n \)
   (7) The columns of \( A \) are a basis for its column space
   (8) If \( Ax = Ay \) then \( x = y \) (uniqueness of solutions to \( Ax = b \))
   (9) The matrix \( A^T A \) is invertible (and symmetric positive definite)
   (10) \( A \) has a left-inverse \( B = (A^T A)^{-1} A^T \), with \( BA = I \)

   Can you find 8 (or 10) parallel statements, all equivalent to this statement 1?

   (1) The rows of \( A \) are independent.

9. Spring 2014, Exam 1, Problem 2
10. Spring 2014, Exam 1, Problem 3
11. Fall 2012, Exam 1, Problem 1
12. 3.5.26
13. 3.6.16
14. A good final practice set is to try Exam1 from Fall 2014. Remember you only have 50min to do it.