Review Questions for Exam II (2015)

1. (a) Find det $A$ by row operations.
(b) Find det $A$ by cofactors of row 1.
(c) Find det $A$ from the Big Formula (how many nonzero terms out of $5! = 120$ possible terms?)

$$
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
$$

2. Suppose $A = BR$ where $R$ is a square invertible matrix.
   (a) Show that $A$ has the same column space as $B$.
   (b) Why does $A(A^T A)^{-1} A^T = B(B^T B)^{-1} B^T$?
   (c) Do $Ax = b$ and $Bx = b$ have the same least squares solution $x$?

3. If $Q$ is an orthogonal matrix, show that

$$
Qx \cdot Qy = x \cdot y \quad \text{length of } Qx = \text{length of } x \quad \text{All } |\lambda_i| = 1.
$$

4. Project $b$ onto the line through $a$. Project the result back onto the line through $b$.

5. What choices of $c$ minimize the squared distances

$$
||\cos 2x - c \cos x||^2 = \int_0^{2\pi} (\cos 2x - c \cos x)^2 dx?
$$

$$
||f(x) - c \cos x||^2 = \int_0^{2\pi} (f(x) - c \cos x)^2 dx?
$$

6. If $A$ is the incidence matrix of a graph, explain the diagonal and off-diagonal parts of $A^T A$:

- $A^T A$ = (degree matrix) Minus (adjacency matrix)
- diagonal entries tell how many edges meet at each node
- 1 if an edge connects nodes $i$ and $j$
- 0 if no edge connects nodes $i$ and $j$

What is $A^T A$ for the complete 5-node graph with all 10 edges? What are the eigenvalues of this 5 by 5 matrix?

7. In $A = QR$, connecting independent vectors $a_1, \ldots, a_n$ to the orthonormal vectors, $q_1, \ldots, q_n$ from Gram-Schmidt, why is $R$ upper triangular?
Solutions to Review Questions Apr 18.

1. (a) Subtract row 4 from row 5
   row 3 from row 4
   row 2 from row 3
   new row 5 from row 1
   new row 1 from row 2

   This produces I. So det A = 1

(b) Co-factors of row 1. det A = (1) C_{11} + (1) C_{15}

   That co-factor C_{11} is +1 (the 4 by 4 is triangular)
   The co-factor C_{15} is 0 (the 4 by 4 has 2 equal rows)

   So det A = (1) 1 + (1) 0 = 1

(c) One non-zero term in the BIG FORMULA: From the 1's down the diagonal.

   I see 8 non-zero terms going into that co-factor C

   So 9 terms in det A (not an exam-type question)

2. (a) \(A = BR\): Each column of A combines cols. of B
   \(B = AR^{-1}\): Each column of B combines columns of A

   Therefore \(C(A) = C(B)\).

(b) Same projection onto same column space!

   OR simplify \(BR (RTBTR)^{-1} ATB\)

(c) No. The \(x^*\) is different for \(Ax = b\), \(Bx = b\).

   True that the \(A^*\) and the \(B^*\) give the same projection \(P\). But \(P = 100 A = B\)

   \(x\) will be \(\frac{1}{100}\) \(x^*\).
3. Two facts $Q^TQ = I$ and $x - y = x^T y$

Then $Qx \cdot Qy = (Qx)^T Qy = x^T Q^T Q y = x^T y$

Also $Qx \cdot Qx = x^T x$ which means $\|Qx\| = \|x\|$. Same length

Also $Qx = \lambda x$ leads to $\|Qx\| = \|x\|$ so $\lambda = 1$

4. Project $b$ to get $p = \frac{a^T b}{a^T a} a$

Then project this $p$ onto the line through $b$

Result: $\frac{b^T p}{b^T b} b = \frac{(b^T a)(a^T b)}{(a^T a)(b^T b)} b$

\[
\int_0^{2\pi} (\cos 2x - c \cos x)^2 \, dx \text{ is just } A c^2 + 2B c + 2
\]

$A = \int (\cos x)^2 \, dx$ $B = \int (\cos x)(\cos 2x) \, dx$ $K = \int (\cos 2x)^2 \, dx$

Minimize $A c^2 + 2B c + 2$

Derivative is $2Ac + 2B = 0$ $c = -\frac{B}{A}$

$c = -\frac{B}{A} = \frac{2\pi}{\int_0^{2\pi} (\cos x)(\cos 2x) \, dx - \int_0^{2\pi} (\cos x)^2 \, dx}$

cos 2x is orthogonal to cos x so the projection is ZERO.

Same steps for $\int (f(x) - c \cos x)^2 \, dx$ lead to

\[
c = -\frac{B}{A} = \frac{\int f(x) \cos x \, dx}{\int (\cos x)^2 \, dx} = \text{FOURIER COEFFICIENT}
\]
6. This asks you to remember incidence matrices. It is not a quiz-type question. Do an example.

\[
\begin{align*}
A & = \begin{bmatrix}
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
-1 & 0 & 0 & 1 \\
0 & -1 & 0 & 0
\end{bmatrix},
\]

\[
A^T A = \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 2 & 0 \\
-1 & -1 & 0 & 2
\end{bmatrix}
\]

Degrees 3, 3, 2, 2
at nodes 1, 2, 3, 4

Yes edge from 2 to 4
no edge from 3 to 4

For the 10 by 5 example, no missing edges.

\[
A^T A = 5 \times 5
\]

Diagonals = 4 = degrees
off-diagonals = -1 = all edges

Eigenvalues of $A^T A = 5 I - (all \ 1's \ matrix)$

Then $\lambda = 5 - (eigenvalues \ of \ all \ ones \ matrix)$

$= 5 - (5, 5, 5, 1, \ trace) = \boxed{16}$

7. $R$ is upper triangular from the order of steps in Gram-Schmidt.

$q_1$ comes from $a_1$;
$q_2$ combines $a_1$ and $a_2$;
$q_3$ combines $a_1, a_2, a_3$;
$q_4$ combines $q_1, q_2, q_3$

$R$ is triangular.