Your PRINTED Name is:  

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**Grading**

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1. (33 points)

(a) Suppose $A$ has the eigenvalues $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$ with eigenvectors $x_1, x_2, x_3$ in the columns of this $S = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$:

$$S = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

What are the eigenvalues and eigenvectors of the matrix $B = A^9 + I$?

(b) How could you find that matrix $B = A^9 + I$ using the eigenvectors in $S$ and the eigenvalues 1, 0, −1?

(c) Give a reason why the matrix $B$ does have or doesn’t have each of these properties:

i. $B$ is invertible

ii. $B$ is symmetric

iii. trace = $B_{11} + B_{22} + B_{33} = 3$. 

2. (33 points)

(a) Show that \( \lambda_1 = 0 \) is an eigenvalue of \( A \) and find an eigenvector \( x_1 \) with that zero eigenvalue:

\[
A = \begin{bmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{bmatrix}
\]

(b) Find the other eigenvalues \( \lambda_2 \) and \( \lambda_3 \) of this symmetric matrix. Does \( A \) have two more independent eigenvectors \( x_2 \) and \( x_3 \)? Give a reason why or why not. (Not required to find \( x_2 \) and \( x_3 \).)

(c) Suppose \( \frac{du}{dt} = Au \) starts from \( u(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \).

Explain why this \( u(t) \) approaches a steady state \( u(\infty) \) as \( t \to \infty \). You can use the general formula

\[
u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3
\]
or

\[
e^{At} = Se^{\Lambda t}S^{-1}
\]

without putting in all eigenvectors. Find that steady state \( u(\infty) \).
3. (34 points)

(a) If $C$ is any symmetric matrix, show that $e^C$ is a positive definite matrix. We can see that $e^C$ is symmetric — which test will you use to show that $e^C$ is positive definite?

(b) $A$ is a 3 by 3 matrix. Suppose $v_1, v_2, v_3$ are orthonormal eigenvectors (with eigenvalues 1, 2, 3) of the symmetric matrix $A^T A$. Show that $Av_1, Av_2, Av_3$ are orthogonal by rewriting and simplifying $(Av_i)^T (Av_j)$.

(c) For the 3 by 3 matrix $A$ in part (b), find three matrices $U, \Sigma, V$ that go into the Singular Value Decomposition $A = U \Sigma V^T$.

(d) True or False: If $A$ is any symmetric 4 by 4 matrix and $M$ is any invertible 4 by 4 matrix, then $B = M^{-1} AM$ is also symmetric. Give a reason for true or false.