Your PRINTED Name is:  ________________________________

Please CIRCLE your section:

Grading  1:

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R01  T10  26-302  Dmitry Vaintrob
R02  T10  26-322  Francesco Lin
R03  T11  26-302  Dmitry Vaintrob
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R05  T11  26-328  Laszlo Lovasz
R06  T12  36-144  Michael Andrews
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R12  T2pm  36-144  Tanya Khovanova
R13  T2pm  26-322  Jay Shah
R14  T3pm  26-322  Carlos Sauer
ESG  26-328  Gabrielle Stoy

Thank you for taking 18.06! I hope you have a wonderful summer!
EACH PART OF EACH QUESTION IS 5 POINTS.

1. (a) Find the reduced row echelon form $R = \text{rref}(A)$ for this matrix $A$:

\[
A = \begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 
\end{bmatrix}.
\]

(b) Find a basis for the column space $C(A)$.

(c) Find all solutions (and first tell me the conditions on $b_1, b_2, b_3$ for solutions to exist!).

\[
Ax = \begin{bmatrix}
b_1 \\
b_2 \\
b_3 
\end{bmatrix}.
\]
2. (a) What is the 3 by 3 projection matrix $P_a$ onto the line through $a = (2, 1, 2)$?

(b) Suppose $P_v$ is the 3 by 3 projection matrix onto the line through $v = (1, 1, 1)$. Find a basis for the column space of the matrix $A = P_aP_v$ (product of 2 projections)
3. Suppose I give you an orthonormal basis \( q_1, \ldots, q_4 \) for \( \mathbf{R}^4 \) and an orthonormal basis \( z_1, \ldots, z_6 \) for \( \mathbf{R}^6 \). From these you create the 6 by 4 matrix \( A = z_1 q_1^T + z_2 q_2^T \).

(a) Find a basis for the nullspace of \( A \).

(b) Find a particular solution to \( Ax = z_1 \) and find the complete solution.

(c) Find \( A^T A \) and find an eigenvector of \( A^T A \) with \( \lambda = 1 \).
4. Symmetric positive definite matrices $H$ and orthogonal matrices $Q$ are the most important. Here is a great theorem: *Every square invertible matrix $A$ can be factored into $A = HQ$.*

(a) Start from $A = U \Sigma V^T$ (the SVD) and choose $Q = UV^T$. Find the other factor $H$ so that $U \Sigma V^T = HQ$. Why is your $H$ symmetric and why is it positive definite?

(b) Factor this 2 by 2 matrix into $A = U \Sigma V^T$ and then into $A = HQ$:

\[
A = \begin{bmatrix}
1 & 3 \\
-1 & 3 \\
\end{bmatrix} = U \Sigma V^T = HQ
\]
5. (a) Are the vectors $(0, 1, 1), (1, 0, 1), (1, 1, 0)$ independent or dependent?
(b) Suppose $T$ is a linear transformation with input space = output space = $\mathbb{R}^3$. We have a basis $u, v, w$ for $\mathbb{R}^3$ and we know that $T(u) = v + w, T(v) = u + w, T(w) = u + v$. Describe the transformation $T^2$ by finding $T^2(u)$ and $T^2(v)$ and $T^2(w)$.
6. Suppose $A$ is a 3 by 3 matrix with eigenvalues $\lambda = 0, 1, -1$ and corresponding eigenvectors $x_1, x_2, x_3$.

(a) What is the rank of $A$? Describe all vectors in its column space $C(A)$.
(b) How would you solve $\frac{du}{dt} = Au$ with $u(0) = (1, 1, 1)$?
(c) What are the eigenvalues and determinant of $e^A$?
7. (a) Find a 2 by 2 matrix such that

\[
A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{and also} \quad A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}
\]

or say why such a matrix can’t exist.

(b) The columns of this matrix \( H \) are orthogonal but not orthonormal:

\[
H = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
0 & -2 & 1 & 1 \\
0 & 0 & -3 & 1
\end{bmatrix}
\]

Find \( H^{-1} \) by the following procedure. First multiply \( H \) by a diagonal matrix \( D \) that makes the columns orthonormal. Then invert. Then account for the diagonal matrix \( D \) to find the 16 entries of \( H^{-1} \).
8. (a) Factor this symmetric matrix into $A = U^T U$ where $U$ is upper triangular:

$$A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 3
\end{bmatrix}.$$

(b) Show by two different tests that $A$ is symmetric positive definite.

(c) Find and explain an upper bound on the eigenvalues of $A$. Find and explain a (positive) lower bound on those eigenvalues if you know that

$$A^{-1} = \begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}.$$
Scrap Paper