Your PRINTED Name is: ________________________________

Please CIRCLE your section:

Grading

1: ________________________________

2: ________________________________

3: ________________________________

4: ________________________________

5: ________________________________

6: ________________________________

7: ________________________________

8: ________________________________

R01 T10 26-302 Dmitry Vaintrob
R02 T10 26-322 Francesco Lin
R03 T11 26-302 Dmitry Vaintrob
R04 T11 26-322 Francesco Lin
R05 T11 26-328 Laszlo Lovasz
R06 T12 36-144 Michael Andrews
R07 T12 26-302 Netanel Blaier
R08 T12 26-328 Laszlo Lovasz
R09 T1pm 26-302 Sungyoon Kim
R10 T1pm 36-144 Tanya Khovanova
R11 T1pm 26-322 Jay Shah
R12 T2pm 36-144 Tanya Khovanova
R13 T2pm 26-322 Jay Shah
R14 T3pm 26-322 Carlos Sauer
ESG Gabrielle Stoy

Thank you for taking 18.06! I hope you have a wonderful summer!
EACH PART OF EACH QUESTION IS 5 POINTS.

1. (a) Find the reduced row echelon form \( R = \text{rref}(A) \) for this matrix \( A \):

\[
A = \begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & 2
\end{bmatrix}.
\]

**Solution.** We have

\[
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & 2
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The last matrix is the RREF.

(b) Find a basis for the column space \( C(A) \).

**Solution.** We can see that the pivot columns are columns 1 and 3, so these columns from the original matrix form a basis,

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
\]

(c) Find all solutions (and first tell me the conditions on \( b_1, b_2, b_3 \) for solutions to exist!).

\[
Ax = \begin{bmatrix} b_1 \\
b_2 \\
b_3 \end{bmatrix}.
\]

**Solution.** We can see that we need \( b_2 = b_3 \). First, let us find a particular solution. Since \( x_2, x_4 \) are free variables, we can set them to 0, and then we can solve to get

\[
\begin{pmatrix}
b_1 - b_2 \\
0 \\
b_2 \\
0
\end{pmatrix}.
\]

Now, we need a basis for the nullspace, the special solutions. Setting each free variable to 1 and the other to 0, we obtain the special
solutions
\[
\begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
-2 \\
1
\end{pmatrix}
\]

So, the general solutions are given by vectors
\[
\begin{pmatrix}
b_1 - b_2 \\
0 \\
b_2 \\
0
\end{pmatrix} + c_1 \begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix} + c_2 \begin{pmatrix}
0 \\
0 \\
-2 \\
1
\end{pmatrix}
\]
2. (a) What is the 3 by 3 projection matrix $P_a$ onto the line through $a = (2, 1, 2)$?

Solution.

$$P_a = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}$$

(b) Suppose $P_v$ is the 3 by 3 projection matrix onto the line through $v = (1, 1, 1)$. Find a basis for the column space of the matrix $A = P_a P_v$ (product of 2 projections)

Solution. $P_a P_v = P_a v = \frac{5}{9}a$ and so $a \in C(P_a P_v) \subset C(P_a)$. Since $C(P_a)$ is spanned by $a$, a basis for $C(P_a P_v)$ is given by \{a\}.
3. Suppose I give you an orthonormal basis $q_1, \ldots, q_4$ for $\mathbb{R}^4$ and an orthonormal basis $z_1, \ldots, z_6$ for $\mathbb{R}^6$. From these you create the 6 by 4 matrix $A = z_1 q_1^T + z_2 q_2^T$.

(a) Find a basis for the nullspace of $A$.

**Solution.** The matrix has SVD $ZJQ^T$ where $J$ is the 6 by 4 matrix with diagonal entries $(1, 1, 0, 0)$. This means that its nullspace consists of the $q$’s in columns of $Q$ corresponding to zero singular values, which is $q_3, q_4$.

(b) Find a particular solution to $Ax = z_1$ and find the complete solution.

**Solution.** One particular solution to $Ax_1 = q_1$, since $(z_1 q_1^T)q_1 = z_1(q_1 \cdot q_1) = z_1$, by and $(z_2 q_2^T)q_1 = z_2(q_2^T q_1) = z_1(q_2 \cdot q_1) = 0$ by orthonormality of $q_i$. The complete solution is obtained by adding an element of the nullspace, i.e. a linear combination of basis vectors of the nullspace: $q_1 + cq_2 + dq_4$ for scalars $c, d$.

(c) Find $A^T A$ and find an eigenvector of $A^T A$ with $\lambda = 1$.

**Solution.** $A^T A = (q_1 z_1^T + q_2 z_2^T)(z_1 q_1^T + z_2 q_2^T)$. Expanding and reparenthezising gives $A^T A = q_1(z_1^T z_1)q_1^T + q_1(z_1^T z_2)q_2^T + q_2(z_2^T z_2)q_2^T$. In every term, the parenthesized scalar in the middle is a dot product: $z_1 \cdot z_2 = 0$ for the middle two terms and 1 for the first and fourth terms, leaving $A^T A = q_1 q_1^T + q_2 q_2^T$. We see that $A^T A q_1 = q_1(q_1 \cdot q_1) + q_2(q_2 \cdot q_1) = q_1$ and, for the same reason, $A^T A q_2 = q_2$. So $q_1$ and $q_2$ (or any nonzero linear combination) are all eigenvectors with eigenvalue 1.
4. Symmetric positive definite matrices $H$ and orthogonal matrices $Q$ are the most important. Here is a great theorem: *Every square invertible matrix $A$ can be factored into $A = HQ$.*

(a) Start from $A = UΣV^T$ (the SVD) and choose $Q = UV^T$. Find the other factor $H$ so that $UΣV^T = HQ$. Why is your $H$ symmetric and why is it positive definite?

**Solution.** By definition we need $UΣV^T = A = HQ = HUV^T$ so we get by inverting $U$ and $V^T$ (which are orthogonal hence invertible) that $H = UΣU^{-1}$. The last item can also be written as $UΣU^T$ because $U$ is orthogonal. This matrix is symmetric because $H^T = (UΣU^T)^T = UΣ^TU^T = H$ as $Σ$ is diagonal so it is equal to its own transpose. To see that it is positive definite we can use the eigenvalue test: the eigenvalues of $H$ are given by the diagonal elements of $Σ$, i.e. the singular values of $A$. They are all *nonnegative* because they are the eigenvalues of $A^TA$, and they cannot be zero because $A$ is invertible by assumption. Hence the eigenvalues of $H$ are all positive.

(b) Factor this 2 by 2 matrix into $A = UΣV^T$ and then into $A = HQ$:

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} = UΣV^T = HQ$$

**Solution.** We have $A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 18 \end{bmatrix}$ so the singular values are $σ_1 = \sqrt{18} = 3\sqrt{2}$ and $σ_2 = \sqrt{2}$ and the corresponding eigenvectors are $v_1 = (0, 1)$ and $v_2 = (1, 0)$ so that $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We then have

$$u_1 = Av_1/σ_1 = (1/\sqrt{2}, 1/\sqrt{2}) \quad u_2 = Av_2/σ_2 = (1/\sqrt{2}, -1/\sqrt{2}),$$

so the SVD is

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$  

Finally

$$H = UΣU^T = \begin{bmatrix} 2\sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2\sqrt{2} \end{bmatrix} \quad Q = UV^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}. $$
5. (a) Are the vectors $(0, 1, 1), (1, 0, 1), (1, 1, 0)$ independent or dependent?

**Solution.** These vectors are independent. One way to see this is that

$$\det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 2 \neq 0$$

(b) Suppose $T$ is a linear transformation with input space = output space = $\mathbb{R}^3$. We have a basis $u, v, w$ for $\mathbb{R}^3$ and we know that $T(u) = v + w, T(v) = u + w, T(w) = u + v$. Describe the transformation $T^2$ by finding $T^2(u)$ and $T^2(v)$ and $T^2(w)$.

**Solution.** We have

$$T^2(u) = T(v + w) = T(v) + T(w) = 2u + v + w$$

$$T^2(v) = T(u + w) = T(u) + T(w) = u + 2v + w$$

$$T^2(w) = T(u + v) = T(u) + T(v) = u + v + 2w$$

Note that this means that in the basis $(u, v, w)$, the matrix of $T^2$ is

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$
6. Suppose $A$ is a 3 by 3 matrix with eigenvalues $\lambda = 0, 1, -1$ and corresponding eigenvectors $x_1, x_2, x_3$.

(a) What is the rank of $A$? Describe all vectors in its column space $C(A)$.

**Solution.** Vectors $x_1, x_2,$ and $x_3$ are independent. Any vector $y$ in $\mathbb{R}^3$ can be represented as a linear combination of the eigenvectors: $y = ax_1 + bx_2 + cx_3$. Applying $A$ we get $Ay = bx_2 - cx_3$. Thus $x_2$ and $x_3$ form a basis in the column space and the rank of $A$ is 2.

(b) How would you solve $du/dt = Au$ with $u(0) = (1, 1, 1)$?

**Solution.** By the formula $u(t) = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$, where $\lambda_i$ are eigenvalues and $x_i$ the corresponding eigenvectors. We are given $\lambda_i$ and $x_i$, so we can plug them in to get: $u(t) = c_1 e^{0t} x_1 + c_2 e^{t} x_2 + c_3 e^{-t} x_3 = c_1 x_1 + c_2 e^{t} x_2 + c_3 e^{-t} x_3$. To find the coefficients $c_1$, $c_2$, and $c_3$, we need to use the initial conditions, that is to solve the equation: $u(0) = (1, 1, 1) = c_1 x_1 + c_2 x_2 + c_3 x_3$.

(c) What are the eigenvalues and determinant of $e^A$?

**Solution.** The eigenvalues of $e^A$ are the same as the eigenvalues of $e^\Lambda$, where $\Lambda$ is the diagonalization of $A$. Therefore, the eigenvalues of $e^A$ equal $e$ to the power of the eigenvalues of $A$: $e^0 = 1$, $e^1 = e$ and $e^{-1} = 1/e$. The determinant is the product of the eigenvalues and is equal to $1 \cdot e \cdot 1/e = 1$. 

---

8
7. (a) Find a 2 by 2 matrix such that
\[
A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ and also } A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}
\]
or say why such a matrix can’t exist.

**Solution.** \( A = \begin{bmatrix} 1/3 & 1/3 \\ 4/3 & 4/3 \end{bmatrix} \) is the 2 by 2 matrix such that \( A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \) and \( A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \). One way to arrive at \( A \) is to let \( B = \begin{bmatrix} 3/4 & 3/4 \\ 4/4 & 4/4 \end{bmatrix} \) be the matrix which sends the standard basis vectors \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) both to \( \begin{bmatrix} 3 \\ 4 \end{bmatrix} \) and let \( C = \begin{bmatrix} 1/2 & 2/1 \\ 1/2 & 1/2 \end{bmatrix} \) be the change of basis matrix which sends the standard basis vectors to \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \). Then \( A = BC^{-1} \).

(b) The columns of this matrix \( H \) are orthogonal but not orthonormal:
\[
H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}
\]
Find \( H^{-1} \) by the following procedure. First multiply \( H \) by a diagonal matrix \( D \) that makes the columns orthonormal. Then invert. Then account for the diagonal matrix \( D \) to find the 16 entries of \( H^{-1} \).

**Solution.** To normalize the columns of \( H \), we let \( D \) be the diagonal matrix with diagonal entries \( 1/\sqrt{2}, 1/\sqrt{6}, 1/\sqrt{12}, \) and \( 1/2 \), and we multiply \( H \) by \( D \) on the right: \( H' = HD \). Because \( H' \) is an orthogonal matrix, \( H'^{-1} = H'^T \). Then \( H^{-1} = D(HD)^{-1} = DH'^T \).

Computing, we obtain \( H^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/6 & 1/6 & -1/3 & 0 \\ 1/12 & 1/12 & 1/12 & -1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \).
8. (a) Factor this symmetric matrix into \( A = U^T U \) where \( U \) is upper triangular:

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 3
\end{bmatrix}.
\]

**Solution.** By applying row operations we find the factorization \( A = LU \)

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

so that \( L = U^T \).

(b) Show by two different tests that \( A \) is symmetric positive definite.

**Solution.** Unfortunately it is hard to compute the eigenvalues explicitly, but nevertheless one can apply one of these tests:

i. \( A = U^T U \) for \( U \) invertible;

ii. the energy test, \( x^T A x = x U^T U x = \| U x \|^2 \geq 0 \) of \( x \neq 0 \) because \( U \) is invertible;

iii. the pivots of \( A \) are the pivots of \( U \) which are all positive;

iv. the upper left determinants of \( A \) are all 1 hence positive;

v. the eigenvalues satisfy the equation \(- (\lambda^3 - 6\lambda^2 + 5\lambda - 1) \) which cannot be zero for negative \( \lambda \) by checking the signs in the sum.

(c) Find and explain an upper bound on the eigenvalues of \( A \). Find and explain a (positive) lower bound on those eigenvalues if you know that

\[
A^{-1} = \begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}.
\]

**Solution.** The eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) are positive and they sum to the trace, which is 6, so they can be at most 6. The inverses of the eigenvalues \( 1/\lambda_1, 1/\lambda_2, 1/\lambda_3 \) are the eigenvalues of \( A^{-1} \), which has trace 5, so this tells us that each of the \( \lambda_i \) is at least 1/5.