Problem 1. Let $\sigma_{\text{max}}(A)$ be the largest singular value of a matrix $A$. Show that $\sigma_{\text{max}}(A^{-1})\sigma_{\text{max}}(A) \geq 1$ for any square invertible matrix $A$.

Let $A$ be an invertible $n \times n$ square matrix. Then the singular values of $A$ are the square roots of the eigenvalues of $AA^T$ or equivalently $A^TA$. Since $A^{-1}(A^{-1})^T = (A^TA)^{-1}$, the singular values of $A^{-1}$ are the reciprocals of the singular values of $A$. Therefore, if $\sigma_1$ is the largest singular value of $A$ and $\sigma_n$ is the smallest singular value of $A$, $\sigma_{\text{max}}(A^{-1})\sigma_{\text{max}}(A) = \sigma_1/\sigma_n$. The known inequality $\sigma_1 \geq \sigma_n$ then yields the desired inequality $\sigma_{\text{max}}(A^{-1})\sigma_{\text{max}}(A) \geq 1$.

Problem 2. Suppose $A$ has orthogonal columns $w_1, w_2, ..., w_n$ of lengths $\sigma_1, \sigma_2, ..., \sigma_n$. What are $U$, $\Sigma$, and $V$ in the SVD?

The problem is slightly complicated (at least notationally) by allowing some lengths to be zero; in your solution you may assume that the $\sigma_i$ are all nonzero (in that case, $k = n$ below). Suppose that the $w_i$ are vectors in $\mathbb{R}^m$, so that $A$ is a $m \times n$ matrix. Since $A$ has pairwise orthogonal columns, $A^TA$ is a $n \times n$ diagonal matrix with eigenvalues the squares of the lengths of the $w_i$, so the singular values of $A$ are the nonzero $\sigma_i$’s. Let $k$ be the number of singular values. Let $f : \{1, ..., n\} \rightarrow \{1, ..., n\}$ be a permutation such that $\sigma_{f(1)} \geq ... \geq \sigma_{f(n)}$. Then $\Sigma$ is a $m \times n$ matrix with $\Sigma_{ii} = \sigma_{f(i)}$ for $1 \leq i \leq k$ and 0 otherwise. $V$ consists of the eigenvectors of $A^TA$, which are the standard basis vectors $e_i$, and we order the eigenvectors by the size of the corresponding eigenvalue. Therefore, $V$ is a $n \times n$ permutation matrix with $i$th column equal to $e_{f(i)}$. We have $A = U\Sigma$, so $U$ is the $m \times m$ matrix with first $k$ columns $\frac{1}{\sigma_{f(1)}}w_{f(1)}, ..., \frac{1}{\sigma_{f(k)}}w_{f(k)}$, in that order, and the rest of the columns all 0.

If you do not follow the convention that the diagonal entries in $\Sigma$ are listed in decreasing order, then we have a simpler SVD (which we also will accept as a valid solution). Namely, $\Sigma$ is a $m \times n$ matrix with $\Sigma_{ii} = \sigma_i$, $V = I$ the $n \times n$ identity matrix, and $U$ has columns $\frac{1}{\sigma_i}w_i$ (with the zero column if $\sigma_i = 0$).

Problem 3. If $A = QR$ with an orthogonal matrix $Q$, the SVD of $A$ is almost the same as the SVD of $R$. Which of the three matrices $U$, $\Sigma$, $V$ is changed because of $Q$?

We have $A^TA = (QR)^T(QR) = R^TR$ because $Q^TQ = I$. Thus the matrices $\Sigma$ and $V$ are unchanged. We have $AA^T = Q(RR^T)Q^T$, so if $u$ is an eigenvector for $AA^T$, then $Q^Tu$ is an eigenvector for $RR^T$. Thus the matrix $U$ becomes $Q^TU$.

Problem 4. Let $n > 1$. Show that there is no $n \times n$ matrix $A$ such that $AM = M^T$ for every $n \times n$ matrix $M$.

Let $M = I$. Then $AM = M^T$ forces $A = I$. However, we do not have $M = M^T$ for every matrix $M$, if $n > 1$. Therefore, no such $A$ exists.
Problem 5. Let $V$ be a vector space and $T : V \rightarrow V$ a linear transformation. Suppose that for every linear transformation $S : V \rightarrow V$, we have $S(T(v)) = T(S(v))$ for all vectors $v \in V$. Show that there exists a scalar $c$ such that $T(v) = cv$ for all vectors $v \in V$.

Let $v_1, ..., v_n$ be a basis for $V$. For any vector $v \in V$, we may write $v = \sum_{i=1}^{n} a_i v_i$ for some $a_i \in \mathbb{R}$. Let's apply this to the vectors $T(v_i)$ to write $T(v_i) = a_{1i} v_1 + ... + a_{ni} v_n$.

Let $S_i : V \rightarrow V$ be the linear transformation defined by $S_i(v_i) = v_i$ and $S_i(v_j) = 0$ for $j \neq i$. Then for every $i$, the given equality $S_i(T(v_i)) = T(S_i(v_i))$ implies that $a_{ii} v_i = T(v_i) = a_{1i} v_1 + ... + a_{ni} v_n$, so $a_{ji} = 0$ for $j \neq i$. It remains to see that $a_{ii} = a_{jj}$ for all $i, j$. For this, let $S_{ij} : V \rightarrow V$ be the linear transformation defined by $S_{ij}(v_i) = v_j$, $S_{ij}(v_j) = v_i$, and $S_{ij}(v_k) = v_k$ for $k \neq i, j$ (actually, $S_{ij}(v_k)$ could be any vector for our purposes). Then $S_{ij}(T(v_i)) = a_{1i} v_j$ and $T(S_{ij}(v_i)) = a_{jj} v_j$, so $a_{ii} = a_{jj}$. Letting $c = a_{11}$, we conclude that $T(v) = cv$ for all vectors $v \in V$.