18.06 Exam II: The Examining
11 March 2016

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RECITATION: R666

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TOTAL /100
1. **Yay or nay**

For each of the following matrices, answer **yes** or **no**: are they invertible? (You do not have to justify your answer.)

(a) \[
\begin{pmatrix}
1 & 4 \\
2 & 5 \\
3 & 6
\end{pmatrix}
\] **no**

(b) \[
\begin{pmatrix}
5 & 1 \\
25 & 5
\end{pmatrix}
\] **no**

(c) \[
\begin{pmatrix}
1 & -1 \\
6 & 5
\end{pmatrix}
\] **yes**

(d) \[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\] **no**

(e) \[
\begin{pmatrix}
1 & 0 & 6 \\
0 & 7 & 8 \\
0 & 0 & 3
\end{pmatrix}
\] **yes**

(f) \[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\] **yes**

(g) \[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & 5 & 6 & 7 \\
0 & 0 & 0 & 8 \\
0 & 0 & 0 & 9
\end{pmatrix}
\] **no**

(h) \[
\begin{pmatrix}
1 & 3 & 5 & 7 & 9 \\
2 & 7 & 12 & 17 & 22 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 4 & 16 & 0 \\
0 & 0 & 5 & 25 & 45
\end{pmatrix}
\] **yes**

(i) \[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
2 & 2 & 2 & 1 & 1 & 2 \\
2 & 2 & 2 & 1 & 2 & 3 \\
2 & 2 & 2 & 2 & 3 & 5
\end{pmatrix}
\] **no**

(j) \[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}
\] **yes**
2. Solve

Find a basis for the space of solutions to the following system of linear equations in the seven variables \(x_1, x_2, x_3, x_4, x_5, x_6, x_7\):

\[
\begin{align*}
    x_1 + x_2 + x_3 + x_4 &= 0 \\
    x_2 + x_3 + x_4 + x_5 &= 0 \\
    x_3 + x_4 + x_5 + x_6 &= 0 \\
    x_4 + x_5 + x_6 + x_7 &= 0
\end{align*}
\]

Solution. We wish to compute the kernel of the matrix

\[
\begin{pmatrix}
    1 & 1 & 1 & 1 & 0 & 0 & 0 \\
    0 & 1 & 1 & 1 & 1 & 0 & 0 \\
    0 & 0 & 1 & 1 & 1 & 1 & 0 \\
    0 & 0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

It’s almost in reduced row echelon form already. Clearing out the 1s over the pivots gives us

\[
\begin{pmatrix}
    1 & 0 & 0 & 0 & -1 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & -1 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & -1 \\
    0 & 0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

So if we set \(x_5 = s, x_6 = t,\) and \(x_7 = u,\) we can write everything in terms of \(s, \ t,\) and \(u:\)

\[
\begin{align*}
    x_1 &= s \\
    x_2 &= t \\
    x_3 &= u \\
    x_4 &= -s - t - u \\
    x_5 &= s \\
    x_6 &= t \\
    x_7 &= u.
\end{align*}
\]

Thus our basis is

\[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.
\]

\(\square\)
3. Rank and file

Compute the rank of the following matrix:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7 \\
5 & 6 & 7 & 8 \\
6 & 7 & 8 & 9 \\
7 & 8 & 9 & 10
\end{pmatrix}
\]

Solution. The rank is the dimension of the span of the column vectors \(\vec{A}^1\), \(\vec{A}^2\), \(\vec{A}^3\), and \(\vec{A}^4\). Simple arithmetic reveals that

\[
\vec{A}^1 + \vec{A}^3 = 2\vec{A}^2
\]

and

\[
\vec{A}^2 + \vec{A}^4 = 2\vec{A}^3.
\]

So, the columns can all be expressed as a linear combination of \(\vec{A}^1\) and \(\vec{A}^2\), and it is obvious that they are not multiples of each other. Hence the column space is 2-dimensional; that is, the rank is 2. \(\square\)
4. **Null at tea**

Find a basis for the kernel of the following matrix:

\[
\begin{pmatrix}
1 & 1 & 2 & 5 & 14 & 42 \\
1 & 2 & 5 & 14 & 42 & 132 \\
2 & 5 & 14 & 42 & 132 & 429
\end{pmatrix}
\]

**Solution.** Just for kicks, let’s use column operations to get the kernel. We’ll clear out each row of the top of the augmented matrix:

\[
\begin{pmatrix}
1 & 1 & 2 & 5 & 14 & 42 \\
1 & 2 & 5 & 14 & 42 & 132 \\
2 & 5 & 14 & 42 & 132 & 429 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 5 & 20 & 75 \\
1 & −1 & 1 & 4 & 14 & 48 \\
0 & 1 & −3 & −9 & −28 & −90 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 0 & 0 & 0 \\
1 & −1 & 1 & −1 & −6 & −27 \\
0 & 1 & −3 & 6 & 32 & 135 \\
0 & 0 & 1 & −5 & −20 & −75 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

The first three columns on the top are a basis of the image, and the last three columns on the bottom are a basis for the kernel:

\[
\begin{bmatrix}
\begin{pmatrix}
−1 \\
6 \\
−5 \\
1 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
−6 \\
32 \\
−20 \\
0 \\
1 \\
0
\end{pmatrix},
\begin{pmatrix}
−27 \\
135 \\
−75 \\
0 \\
0 \\
1
\end{pmatrix}
\end{bmatrix}
\]

\[\square\]
5. Inversion invasion

Compute the inverse of this matrix:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 1 & 3 & 6 & 10 \\
0 & 0 & 0 & 1 & 4 & 10 \\
0 & 0 & 0 & 0 & 1 & 5 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

**Solution.** Everyone’s favorite way to invert a matrix is with row operations applied to \((A|I)\) (where \(A\) is our matrix):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 | 1 & -1 & 1 & -1 & 1 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 | 0 & 1 & -2 & 3 & -4 & 5 \\
0 & 0 & 1 & 3 & 6 & 10 | 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 4 & 10 | 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 5 | 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 | 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 | 1 & -1 & 1 & -1 & 1 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 | 0 & 1 & -2 & 3 & -4 & 5 \\
0 & 0 & 1 & 3 & 6 & 10 | 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 4 & 10 | 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 5 | 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 | 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 | 1 & -1 & 1 & -1 & 1 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 | 0 & 1 & -2 & 3 & -4 & 5 \\
0 & 0 & 1 & 0 & 0 & 0 | 0 & 1 & -3 & 6 & -10 \\
0 & 0 & 0 & 1 & 4 & 10 | 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 5 | 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 | 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 | 1 & -1 & 1 & -1 & 1 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 | 0 & 1 & -2 & 3 & -4 & 5 \\
0 & 0 & 1 & 0 & 0 & 0 | 0 & 1 & -3 & 6 & -10 \\
0 & 0 & 0 & 1 & 0 & 0 | 0 & 1 & -4 & 10 \\
0 & 0 & 0 & 0 & 1 & 5 | 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 | 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
So, magically,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -1 & 1 & -1 & 1 & -1 \\
0 & 1 & -2 & 3 & -4 & 5 \\
0 & 0 & 1 & -3 & 6 & -10 \\
0 & 0 & 0 & 1 & -4 & 10 \\
0 & 0 & 0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 1 & 3 & 6 & 10 \\
0 & 0 & 0 & 1 & 4 & 10 \\
0 & 0 & 0 & 0 & 1 & 5 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
1 & -1 & 1 & -1 & 1 & -1 \\
0 & 1 & -2 & 3 & -4 & 5 \\
0 & 0 & 1 & -3 & 6 & -10 \\
0 & 0 & 0 & 1 & -4 & 10 \\
0 & 0 & 0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]