18.06.15: ‘Rank-nullity: the sequel’

Lecturer: Barwick

Wednesday 9 March 2016
Let’s take a moment to imagine how our proof of the Rank-Nullity Theorem might have been different if we’d used column operations to get our matrix into rcef:

\[ \text{column operations: } A \xrightarrow{\text{\ldots}} AN, \]

where \( N \) is an invertible \( n \times n \) matrix.
The point here is that *column operations don’t change the image*:

\[ \text{im}(A) = \text{im}(AN). \]

However, *column operations absolutely do change the kernel*:

\[ \ker(A) \neq \ker(AN). \]

**BUT,** *column operations don’t change the *dimension* of the kernel*:

\[ \dim(\ker(A)) = \dim(\ker(AN)). \]
Using pure thought, tell me what the rank and nullity are of these matrices:

\[
\begin{pmatrix}
5 & -15 \\
-2 & 6
\end{pmatrix}
\]

\[
\begin{pmatrix}
2 & 4 & -138 \\
5 & 1 & 75
\end{pmatrix}
\]

\[
\begin{pmatrix}
2 & 6 & 3 \\
5 & 1 & 50 \\
0 & 0 & 0
\end{pmatrix}
\]
18.06.15: ‘Rank-nullity: the sequel’

\[
\begin{pmatrix}
9 & 9 & 9 \\
1 & 1 & 1 \\
4 & 4 & 4 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 5 & 7 \\
-2 & 6 & 3 \\
-1 & 11 & 10 \\
\end{pmatrix}
\]
18.06.15: ‘Rank-nullity: the sequel’

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
1 & 2 & 4 & 8 & 16
\end{pmatrix}
\]
One final topic that we didn’t yet discuss. We’ve focused on solving equations $A\vec{x} = \vec{0}$, but what about the more general equation $A\vec{x} = \vec{v}$? What do we do there?

There are two options:

- **no solutions** – here $\vec{v}$ does not lie in the image of $A$;
- **at least one solution** – here $\vec{v} \in \text{im}(A)$.

In the latter case, let’s try to work out a way to find all the solutions to $A\vec{x} = \vec{v}$. 
Let’s suppose we’ve located one solution – a vector $\vec{x}_0 \in \mathbb{R}^n$ such that $A\vec{x}_0 = \vec{v}$. It turns out we can get all of them from that one, if we know about ... the kernel!

Why? Well, suppose $\vec{y} \in \ker(A)$. Then

$$A(\vec{x}_0 + \vec{y}) = A\vec{x}_0 + A\vec{y} = \vec{v} + \vec{0} = \vec{v}.$$ 

On the other hand, if $A\vec{x} = \vec{v}$, then

$$A(\vec{x} - \vec{x}_0) = A\vec{x} - A\vec{x}_0 = \vec{v} - \vec{v} = \vec{0}.$$
Hence the set of solutions to the equation $A\vec{x} = \vec{v}$ is the set

$$\{\vec{x} = \vec{x}_0 + \vec{y} \mid \vec{y} \in \ker(A)\}.$$
Let’s do this in an example. If we have

\[ \vec{v} = \begin{pmatrix} -4 \\ 3 \\ 7 \end{pmatrix} \]

and

\[ A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}, \]

let’s find the set of solutions \( \vec{x} \) to the equation \( A\vec{x} = \vec{v} \).
The first step is to find a basis of \( \ker(A) \):

\[
\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}
\]
On the other hand, it’s easy to find one solution:

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix}.
\]
So, any solution can be written in a unique fashion as

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
+ s
\begin{pmatrix}
2 \\
1 \\
0 \\
0
\end{pmatrix}
+ t
\begin{pmatrix}
1 \\
0 \\
-2 \\
1
\end{pmatrix}
+ u
\begin{pmatrix}
-3 \\
0 \\
2 \\
0
\end{pmatrix}.
\]