18.06.16: The four fundamental subspaces

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Here it is again:

**Theorem** (Rank-Nullity Theorem). *If $A$ is an $m \times n$ matrix, then*

$$\dim(\ker(A)) + \dim(\operatorname{im}(A)) = n.$$  

We saw a proof of this by reducing to rref or rcef, and then checking it there. There’s just one thing that might bug us here. If I think of the linear map

$$T_A: \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$

then we see that $\ker(A)$ is a subspace of the source $\mathbb{R}^n$, but $\operatorname{im}(A)$ is a subspace of the target $\mathbb{R}^m$. So why should these spaces be related?
To answer this question, there’s another matrix we can contemplate, the transpose $A^\top$. This is an $n \times m$ matrix, and so it corresponds to a linear map in the other direction:

$$T_{A^\top} : \mathbb{R}^m \longrightarrow \mathbb{R}^n.$$ 

This is the map that takes a column vector $\vec{v}$ and builds the column vector $A^\top \vec{v}$, but we can perform a trick here. Instead of thinking about transposing $A$, we can think about transposing the vectors.
So \((\mathbb{R}^m)^\vee\) will be the set of all \(m\)-dimensional row vectors; equivalently, the set of all \(1 \times m\) matrices; equivalently again, the set of all transposes

\[ \vec{v} := (\vec{v})^\top \]

of vectors \(\vec{v} \in \mathbb{R}^m\). We call \((\mathbb{R}^m)^\vee\) the dual \(\mathbb{R}^m\).

So, e.g., if \(\vec{v} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}\), then \(\vec{v} = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \end{pmatrix}\).
The neat thing about row vectors is that they *do stuff* to column vectors. If \( \vec{v} \in \mathbb{R}^n \) and \( \vec{w} \in (\mathbb{R}^n)^\vee \), then \( \vec{w} \vec{v} \) is a *number*. (Question: How is this related to the dot product?)

If \( V \subseteq \mathbb{R}^n \) is a vector subspace, then we define

\[
V^\perp := \{ \vec{w} \in (\mathbb{R}^n)^\vee \mid \text{for any } \vec{v} \in V, \vec{w} \vec{v} = 0 \} \subseteq (\mathbb{R}^n)^\vee .
\]

Dually, if \( W \subseteq (\mathbb{R}^n)^\vee \) is a vector subspace, then we define

\[
W^\perp := \{ \vec{v} \in \mathbb{R}^n \mid \text{for any } \vec{w} \in W, \vec{w} \vec{v} = 0 \} \subseteq \mathbb{R}^n .
\]

Fact: \( \dim(V) = n - \dim(V^\perp) \), and \( \dim(W) = n - \dim(W^\perp) \). (Why?)
Now, since

\[(A^\top \tilde{v})^\top = (\tilde{v})^\top A = \nu A,\]

we can leave $A$ just as it is, and we can consider the linear map

\[T_A^\vee : (\mathbb{R}^m)^\vee \longrightarrow (\mathbb{R}^n)^\vee\]

given by the formula

\[T_A^\vee (\nu) := \nu A.\]
So when we contemplate the kernel and image of $A^\top$, we can think about it via the map $T^\vee_A$.

For example, $\ker(A^\top)$ is the set of all vectors $\underline{\nu} \in (\mathbb{R}^n)^\vee$ such that $\underline{\nu} A = \underline{0}$. This space is also called the *cokernel* or the *left kernel* of $A$. I write $\text{coker}(A)$.

We also have the image of $A^\top$, which is the set of all row vectors $\underline{\omega} \in (\mathbb{R}^n)^\vee$ such that there exists a row vector $\underline{\nu} \in (\mathbb{R}^m)^\vee$ for which $\underline{\omega} = \underline{\nu} A$. This space is also called the *coimage* of $A$, or, since its the span of the columns of $A^\top$, which is the span of the rows of $A$, it is also called the *row space* of $A$. I write $\text{coim}(A)$.
In all, we have four vector spaces that are what Strang call the *fundamental subspaces* attached to $A$:

$$\text{ker}(A), \quad \text{im}(A), \quad \text{coker}(A) := \ker(A^T), \quad \text{coim}(A) := \text{im}(A^T).$$
Here’s the abstract statement of the Rank-Nullity Theorem:

(1) \( \ker(A) = \text{coim}(A) \perp \), so that

\[
\dim(\ker(A)) = n - \dim(\text{coim}(A)).
\]

(2) \( \text{im}(A) = \text{coker}(A) \perp \), so that

\[
\dim(\text{im}(A)) = m - \dim(\text{coker}(A)).
\]

(3) \( A \) provides a bijection \( \text{coim}(A) \cong \text{im}(A) \), so that

\[
\dim(\text{coim}(A)) = \dim(\text{im}(A)).
\]