18.06.30: Spectral theorem

Lecturer: Barwick

And in the telling of that story
I lose my way inside a prepositional phrase.
– Wye Oak
For \( v, w \in \mathbb{C}^n \), write

\[
\langle v \mid w \rangle := v^* w.
\]

This is a complex number, called the *inner product* of two complex vectors; it extends the usual dot product, but notices that the linearity in the first coordinate is *twisted*:

\[
\langle \alpha v \mid w \rangle = \overline{\alpha} \langle v \mid w \rangle \quad \text{but} \quad \langle v \mid \alpha w \rangle = \alpha \langle v \mid w \rangle.
\]

The length of a vector \( v \in \mathbb{C}^n \) is defined by \( ||v||^2 = \langle v \mid v \rangle \); it’s precisely the same as the length of the corresponding vector in \( \mathbb{R}^{2n} \). (Why??)
Lemma. An $n \times n$ complex matrix $B$ is Hermitian if and only if, for any $v, w \in \mathbb{C}^n$,

$$
\langle Av | w \rangle = \langle v | Aw \rangle.
$$

Proof. If $A$ is Hermitian, then $(Av)^* w = v^* A^* w = v^* Aw$.

On the other hand, suppose that for any $v, w \in \mathbb{C}^n$,

$$
\langle Av | w \rangle = \langle v | Aw \rangle.
$$

Then when $v = \hat{e}_i$ and $w = \hat{e}_j$, this equation becomes

$$
\bar{a}_{ji} = (A^i)^* \hat{e}_j = \hat{e}_i^* A^j = a_{ij}.
$$
Theorem (Spectral theorem; last big result of the semester). Suppose $B$ a Hermitian matrix. Then

1. The eigenvalues of $B$ are real.

2. There is an orthogonal basis of eigenvectors for $B$; in particular, $B$ is diagonalizable over $\mathbb{C}$ (and even over $\mathbb{R}$ if $B$ has real entries).
Proof. Let’s first see why the eigenvalues of $B$ must be real. Suppose $v \in \mathbb{C}^n$ an eigenvector of $B$ with eigenvalue $\lambda$, so that $Bv = \lambda v$. Then,

$$
\lambda \|v\|^2 = \lambda \langle v | v \rangle = \langle v | \lambda v \rangle \\
= \langle v | Bv \rangle \\
= \langle Bv | v \rangle \\
= \langle \lambda v | v \rangle \\
= \overline{\lambda} \langle v | v \rangle = \overline{\lambda} \|v\|^2.
$$

Since $v \neq 0$, one has $\|v\| \neq 0$, whence $\lambda = \overline{\lambda}$. 
Now let’s see about that orthogonal basis of eigenvectors. Using the Fundamental Theorem of Algebra, write the characteristic polynomial

\[ p_B(t) = (t - \lambda_1) \cdots (t - \lambda_n), \]

where \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) are the roots of \( p_B \). We may not assume that the \( \lambda_i \)'s are distinct!!

Let’s choose an eigenvector \( v_1 \) with eigenvalue \( \lambda_1 \), and consider the hyperplane

\[ W_1 := \{ w \in \mathbb{C}^n \mid \langle v_1 | w \rangle = 0 \}. \]
Note that for any \( w \in W_1 \), one has

\[
\langle v_1 | Bw \rangle = \langle Bv_1 | w \rangle = \langle \lambda_1 v_1 | Bw \rangle = \lambda_1 \langle v_1 | w \rangle = 0,
\]

so \( Bw \in W_1 \) as well. Hence the linear map \( T_B : C^n \rightarrow C^n \) restricts to a map \( T_1 : W_1 \rightarrow W_1 \).
Select, temporarily, a $\mathbb{C}$-basis $\{w_2, \ldots, w_n\}$ of $W_1$. Then $\{v_1, w_2, \ldots, w_n\}$ is a $\mathbb{C}$-basis of $\mathbb{C}^n$, and writing $T_B$ relative to this basis gives us a matrix

$$C_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & B_1 \end{pmatrix},$$

where $B_1$ is the $(n - 1) \times (n - 1)$ matrix that represents $T_1$ relative to the basis $\{w_2, \ldots, w_n\}$, and

$$p_{B_1} = (t - \lambda_2) \cdots (t - \lambda_n).$$
Now we run that same argument again with the $(n - 1) \times (n - 1)$ matrix $B_1$ in place of the $n \times n$ matrix $B$ to get:

- an eigenvector $v_2 \in W_1$ with eigenvalue $\lambda_2$,
- the subspace $W_2 \subset W_1$ of vectors orthogonal to $v_2$,
- and an $(n - 2) \times (n - 2)$ matrix $B_2$ that represents $T_B$ restricted to $W_2$.

Now we find that $B$ is similar to

$$C_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & B_2 \end{pmatrix}.$$
We repeat this argument repeatedly on each new $B_i$, each time getting:

- an eigenvector $v_{i+1} \in W_i$ with eigenvalue $\lambda_{i+1}$,
- the subspace $W_{i+1} \subset W_i$ of vectors orthogonal to $v_{i+1}$,
- and an $(n - i - 1) \times (n - i - 1)$ matrix $B_{i+1}$ that represents $T_B$ restricted to $W_{i+1}$. 
At each stage, we find that $B$ is similar to

$$
C_{i+1} = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 & 0 \\
0 & \lambda_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda_i & 0 \\
0 & 0 & \cdots & 0 & B_{i+1}
\end{pmatrix}.
$$
This process eventually stops, when \( i = n \). Then we’re left with:

- eigenvectors \( v_1, \ldots, v_n \) with eigenvalues \( \lambda_1, \ldots, \lambda_n \),

- a string of subspaces

\[
C^n = W_0 \supset W_1 \supset W_2 \supset \cdots \supset W_n = \{0\},
\]

with \( v_{i+1} \in W_i \), and

\[
W_{i+1} = \{ w \in W_i \mid \langle v_{i+1} | w \rangle \},
\]

- and a diagonal matrix \( C_n = \text{diag}(\lambda_1, \ldots, \lambda_n) \) to which \( B \) is similar.