Problem 1

(a) $A$ is invertible, so $\ker A = \{0\}$.

(b), (c) and so on:

Let us solve the $n \times n$ case for $n \geq 3$. Let $c_j$ be the $j$-th column of our matrix $A$. A direct computation shows that

$$c_2 - c_1 = c_3 - c_2 = \cdots = c_n - c_{n-1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

hence $c_1 - 2c_2 + c_3 = c_2 - 2c_3 + c_4 = \cdots = c_{n-2} - 2c_{n-1} + c_n = 0$. From these linear relations between columns of $A$, we can extract $n - 2$ solutions to the equation $A\vec{x} = \vec{0}$. They are

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \vec{x}_{n-2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -2 \end{pmatrix}.$$

It is not hard to check that $\vec{x}_1, \ldots, \vec{x}_{n-2}$ are linearly independent vectors, so

(1) $\dim(\ker A) \geq n - 2.$

On the other hand, notice that the first 2 columns of $A$ are linearly independent, therefore $\dim(\text{im } A) \geq 2$. By the Rank-Nullity Theorem (see Lecture 13 slides), we know that

(2) $\dim(\ker A) = n - \dim(\text{im } A) \leq n - 2.$

Compare (1) and (2) we conclude that $\dim(\ker A) = n - 2$ and therefore $\{\vec{x}_1, \ldots, \vec{x}_{n-2}\}$ forms a basis of $\ker A$.

Remark 0.1. What described above is exactly the column operation method covered in Lecture 13.

Problem 2

Both the method and answer are identical to Problem 1.
Problem 3

If we write down the matrix $A$, it is easy to find that

\[
c_1 = c_3 = c_5 = \cdots = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{pmatrix}
\]

and

\[
c_2 = c_4 = c_6 = \cdots = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}.
\]

Use the argument in Problem 1, one conclude that

\[
\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \end{pmatrix}
\]

is a basis of $\text{ker } A$. There are $(n - 2)$ of them.

Problem 4

Suppose \( \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) is a vector in $\text{ker } X$. It is equivalent to say that

\[
\begin{cases}
Av_1 + Bv_2 = 0 \\
Cv_2 = 0
\end{cases}
\]

Hence for any $v_1 \in \text{ker } A$, the vector $\begin{pmatrix} v_1 \\ 0 \end{pmatrix}$ lives in $\text{ker } X$, i.e.,

\[
\begin{pmatrix} \text{ker } A \\ 0 \end{pmatrix} \subset \text{ker } X,
\]

in particular

\[
\dim \text{ker } X \geq \dim \ker A.
\]

On the other hand, for $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \text{ker } X$, we know that $v_2 \in \text{ker } C$. Moreover, fix $v_2 \in \text{ker } C$, the solutions to

\[
Av_1 = -Bv_2
\]
is either the empty set or an affine space of dimension \(\dim \ker A\) (meaning that the difference of any two solutions \(v_1\) and \(v_1'\) is a vector in \(\ker A\)), so we conclude that

\[
\dim \ker X \leq \dim \ker A + \dim \ker C.
\]

Combining two parts, we see that

(3) \[
\dim \ker A \leq \dim \ker X \leq \dim \ker A + \dim \ker C.
\]

Remark 0.2. Maybe a better way to reach the final result is to observe that

\[
q + \text{column rank of } A \geq \text{column rank of } X \geq \text{column rank of } C + \text{column rank of } A,
\]

and deduce the final result from the Rank-Nullity Theorem.

Problem 5

Direct computation shows that

\[
Q^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},
\]

hence \(Q^2 - Q - I = 0\), where \(I\) is the 2 by 2 identity matrix. Use the relation \(Q^2 = Q + I\) repetitively, we see that

\[
Q^{-1} = Q^{-1} \cdot I = Q^{-1}(Q^2 - Q) = Q - I = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},
\]

\[
Q^{-2} = (Q^{-1})^2 = (Q - I)^2 = Q^2 - 2Q + I = 2I - Q = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},
\]

\[
Q^{-3} = Q^{-2}Q^{-1} = (2I - Q)(Q - I) = -Q^2 + 3Q - 2I = 2Q - 3I = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix},
\]

\[
Q^3 = Q^2Q = (Q + I)Q = Q^2 + Q = 2Q + I = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}.
\]

In general, we have

\[
Q^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}
\]

and we can deduce this by induction on \(n\), using the fact that

\[
Q^n = Q^{n-2}Q^2 = Q^{n-2}(Q + I) = Q^{n-1} + Q^{n-2}.
\]

Taking the determinant of \(Q^n\), we see that

\[
f_{n+1}f_{n-1} - f_n^2 = \det Q^n = (\det Q)^n = (-1)^n,
\]

therefore

\[
f_n^2 + (-1)^n = f_{n-1}f_{n+1}.
\]

There are many things we can do by playing with \(Q\). For instance, consider

\[
Q^{n+m} = Q^nQ^m = Q^mQ^n,
\]

then we can write down

\[
\begin{pmatrix} f_{n+m+1} & f_{n+m} \\ f_{n+m} & f_{n+m-1} \end{pmatrix} = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \begin{pmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{pmatrix}.
\]
Compare the $(1,1)$-entry, we see that
\[ f_{n+m+1} = f_{n+1}f_{m+1} + f_n f_m. \]

In particular, if we choose $n = m$, we see that
\[ f_{2n+1} = f_{n+1}^2 + f_n^2. \]

Therefore $f_{2n+1}$ can be expressed as a sum of two squares, hence (if you know some number theory) $f_{2n+1}$ is not a multiple of 3, 7, 11, 19, 23, \ldots.

**Problem 6**

Let $c_j$ be the $j$-th column of our matrix $F$. By the definition of Fibonacci numbers, we see that
\[ c_1 + c_2 - c_3 = c_2 + c_3 - c_4 = \cdots = c_{n-2} + c_{n-1} - c_n = 0. \]

Use the method in Problem 1, we find that a basis of ker $F$ is given by
\[
\begin{pmatrix}
1 \\
1 \\
-1 \\
0 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
1 \\
-1 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}, \ldots,
\begin{pmatrix}
0 \\
0 \\
1 \\
-1
\end{pmatrix}.
\]

There are $(n - 2)$ of them.