(1) Using only the identities $e^{i\theta} = \cos \theta + i \sin \theta$ and $\cos^2 \theta + \sin^2 \theta = 1$ along with the basic functioning of the complex numbers, prove the following trigonometric identities:

(a) $\cos(\alpha \pm \beta) = (\cos \alpha)(\cos \beta) \mp (\sin \alpha)(\sin \beta)$;

(b) $\sin(\alpha \pm \beta) = (\sin \alpha)(\cos \beta) \pm (\cos \alpha)(\sin \beta)$;

(c) $\sin^2 \alpha = \frac{1}{2}(1 - \cos(2\alpha))$;

(d) $\cos^2 \alpha = \frac{1}{2}(1 + \cos(2\alpha))$.

(In the same way, you can also prove a formula for $\cos^n$ and $\sin^n$ in terms of only sines and cosines.)
(2) For any complex number \( z = a + bi \), consider the matrix
\[
M_z := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.
\]

Prove that these matrices contain all the algebraic properties of \( \mathbb{C} \) by verifying the following:

(a) \( M_0 = 0 \) and \( M_1 = I \);  
(b) \( M_{z+w} = M_z + M_w \) for any \( z, w \in \mathbb{C} \);  
(c) \( M_{-z} = -M_z \) for any \( z \in \mathbb{C} \);  
(d) \( M_{zw} = M_z M_w \) for any \( z, w \in \mathbb{C} \);  
(e) \( M_{z^{-1}} = M_z^{-1} \) for any \( z \in \mathbb{C} \) such that \( z \neq 0 \);  
(f) \( M_{\overline{z}} = M_{\overline{z}}^T \) for any \( z \in \mathbb{C} \);  
(g) \( |z|^2 = \det(M_z) \) for any \( z \in \mathbb{C} \);  
(h) \( t^2 - (z + \overline{z})t + z\overline{z} = p_{M_z}(t) \) for any \( z \in \mathbb{C} \);  
(i) \( M_{\rho \exp(i\theta)} = \rho \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \) for any \( \rho \geq 0 \) and any \( \theta \in [0, 2\pi) \).
(3) Consider the $n \times n$ matrix

$$A = (\hat{e}_2 \; \cdots \; \hat{e}_n \; \hat{e}_1).$$

Compute the characteristic polynomial and the complex eigenvalues of $A$. Is $A$ diagonalizable over $\mathbb{R}$? over $\mathbb{C}$?
(4) Suppose \( \theta \in [0, 2\pi) \). What are the complex eigenvalues and corresponding complex eigenspaces of the matrix

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]
(5) Suppose $\mathbf{x} \in \mathbb{C}^n$ a vector such that $\mathbf{x}^* \mathbf{x} = 1$. What are the eigenspaces of $I - 2\mathbf{x}\mathbf{x}^*$?
(6) Suppose \( A \) is an \( n \times n \) matrix with characteristic polynomial

\[
p_A(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 x + a_0.
\]

Find an expression for \( p_{A^{-1}}(t) \) by contemplating the determinant of \((tI - A^{-1})A\).
(7) For any $n \geq 2$, give an example of an invertible $n \times n$ matrix that is not diagonalizable over $\mathbb{C}$. 