1. (a) \(|A| = 2\) times the 3 by 3 determinant = 2 times 0 = 0.

(b) \(A\) has rank 3 so we want three orthogonal basis vectors \(A, B, C\):

\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
B = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

To orthogonalize divide by lengths:

\[
q_1 = \frac{A}{\|A\|} = \frac{A}{\sqrt{3}}
\]

\[
= \frac{(1, 1, 1, 0)}{\sqrt{3}}
\]

\[
q_2 = \frac{B}{2\sqrt{2}}
\]

\[
= \frac{(2, 2, 2, 0)}{2\sqrt{2}}
\]

\[
q_3 = \frac{C}{2}
\]

\[
= \frac{(0, 0, 0)}{2}
\]

(c) Adding 1 to the \(a_{11}\) entry will add its cofactor to the determinant:

\[
\text{Cofactor } C_{11} = \begin{vmatrix} 1 & 5 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -12.
\]
2 (a) \[
\begin{bmatrix}
1 & -4 \\
1 & -3 \\
1 & -2 \\
1 & -1 \\
1 & 0 \\
1 & 1 \\
1 & 2 \\
1 & 3 \\
1 & 4
\end{bmatrix}
\begin{bmatrix}
C \\
D
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
10
\end{bmatrix}
\]

(b) \[
A^T A = \begin{bmatrix} 9 & 0 \\ 0 & 60 \end{bmatrix}, \quad A^T b = \begin{bmatrix} 10 \\ 40 \end{bmatrix}
\]

Solve \( A^T A \begin{bmatrix} C \\ D \end{bmatrix} = A^b \) to find \( C = \frac{10}{9}, \ D = \frac{40}{60}. \)

(c) The columns of \( A \) are a basis for the subspace. The projection is
\[
p = C \text{ (column 1)} + D \text{ (column 2)}.
\]

3 (a) \( Q \) has rank \( n \) (the \( n \) orthonormal columns are independent).

(b) \( P = Q (Q^T Q)^{-1} Q^T = QQ^T. \)

(c) Check \( P^T = P: (QQ^T)^T = QQ^T. \)
\[\text{Check } P^2 = P: (QQ^T)Q^T = QQ^T.\]

4 (a) The length of \( \lambda \) is \(|\lambda||x||. \)
\[\text{The length squared of } Qx \text{ is } (Qx)^T (Qx) = x^T Q^T Q x = x^T x = x^T x. \]
Thus \(|\lambda||x|| = ||x|| \text{ and } |\lambda| = 1. \]
\[\text{Note: We did not use the correct notation when } \lambda \text{ and } x \text{ are complex. The reasoning stays the same.} \]

(b) Projection onto the last column:
\[
p = \frac{a^T b}{a^T a} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} 0 = \text{zero vector.}
\]

Projection onto column space (which is all of \( R^1 \)) is \( b \) itself.

(c) \[|H_2 - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 2 = 0. \]
The eigenvalues are \( \sqrt{2} \) and \( -\sqrt{2} \). Check trace = 0 and determinant = \(-2. \)