18.099b Problem Set 5a  
Due: Thursday, April 1st (in class or before).

This assignment is to write a short paper on the oscillation of a function. It should be typeset in TeX. Arrange and express the material discussed below in whatever way you think best. Be sure to define all the terms I have italicised (even if I have not defined them here).

Given a function \( f(x) \) and a real number \( c \), we say that \( f \) is bounded about \( c \) if there exists some \( \epsilon > 0 \) such that

(i) \( f \) is defined on \([c - \epsilon, c + \epsilon]\) except possibly at \( c \) itself, and
(ii) the set \( \{f(x) : x \in [c - \epsilon, c + \epsilon], x \neq c\} \) is bounded from above and below.

For any such \( \epsilon \) let \( A_{\epsilon} := \{f(x) : x \in [c - \epsilon, c + \epsilon], x \neq c\} \). The oscillation of \( f \) at \( c \) is defined to be the number

\[
\text{osc}_f(c) := \lim_{\epsilon \to 0} (\sup A_{\epsilon} - \inf A_{\epsilon})
\]

Show that this limit exists and hence \( \text{osc}_f(c) \) is well defined.

Show that for functions \( f \) bounded about a real number \( c \), \( \text{osc}_f(c) = 0 \) if and only if \( f \) is continuous or has a removable singularity at \( c \). Give two examples of non-zero oscillation, one where \( c \) is a jump singularity and one where \( c \) is an essential singularity.

An intuitive way to describe what it means for a function to be continuous at a point \( c \) would be to say: “\( f(x) \) can be made arbitrarily close to \( f(c) \) by making \( x \) arbitrarily close to \( c \)”. Fixing \( \tau \geq 0 \), give an analogous intuitive description of what it means for \( f \) to have oscillation \( \tau \) at a point \( c \). The idea is that oscillation measures the “amount” of discontinuity.

Suppose \( f \) is monotonically increasing and defined on an interval \([a, b]\). Note that \( f \) cannot have any removable singularities in \([a, b]\), and that that for any \( c \in (a, b) \), \( f \) is bounded about \( c \). Moreover, \( \text{osc}_f(c) = \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x) \). Conclude that for any \( \tau > 0 \), the number of points in \((a, b)\) at which \( f \) has oscillation greater than or equal to \( \tau \) is at most \( \frac{f(b) - f(a)}{\tau} \). Use this to show that \( f \) has at most countably many discontinuities in \((a, b)\). (Recall that a set \( S \) is countable if there is a bijection between \( \mathbb{N} \) and \( S \)).