LIMITS OF FUNCTIONS

Given \( f : \mathbb{R} \to \mathbb{R}, \lim_{x \to p} f(x) = q \) means that \(|f(x) - q| < \epsilon\) whenever \(0 < |x - p| < \delta\).

**Theorem 1.** \( \lim_{x \to p} f(x) = q \) if and only if \( \lim_{n \to \infty} f(p_n) = q \) whenever \( \lim_{n \to \infty} p_n = p \) with \( p_n \neq p \).

**Proof.** Suppose \( \lim_{x \to p} f(x) = q \) and let \( \{p_n\} \) be any sequence of real numbers converging to \( p \), but not equal to \( p \). Let \( \epsilon > 0 \) be arbitrary. Then there exists a \( \delta > 0 \) such that \(|f(x) - q| < \epsilon\) if \(0 < |x - p| < \delta\). On the other hand, there is a natural number \( N \) such that for all \( n > N \), \(0 < |p_n - p| < \delta\). Hence for all \( n > N \), \(|f(p_n) - q| < \epsilon\), and so \( \{f(p_n)\} \) converges to \( q \).

Suppose \( \lim_{x \to p} f(x) \neq q \). Then there exists some \( \epsilon > 0 \) such that for every \( \delta > 0 \) there exists some \( x(\delta) \in \mathbb{R} \) for which \(0 < |x(\delta) - p| < \delta\) but \(|f(x(\delta)) - q| \geq \epsilon\). Hence

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\lim_{n \to \infty} f(x(\frac{1}{n})) \neq q \quad \text{but} \quad \lim_{n \to \infty} x(\frac{1}{n}) = p. \quad \blacksquare
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