From Gaussian Ensembles to Stochastic Operators

1. (Real) GOE "Gaussian orthogonal Ensemble"
   \[ A = \text{randn}(n) \]
   \[ S = (A + A^\dagger) / \sqrt{2n} \]

   Fact: \( \lambda_{\text{max}}(S) \approx 2 \), \( \lambda_{\text{min}}(S) \approx -2 \) when \( n \to \infty \)

   Fact: Normalized Histogram \( \approx \frac{1}{2\pi \sqrt{4-x^2}} \)
   \( (A + A^\dagger) / 2 \) has \( \mathcal{N}(0,1) \) on diagonal \( + \mathcal{N}(0, \frac{1}{2}) \) off

2. (Complex) GUE "Gaussian Unitary Ensemble"
   \[ A = \text{randn}(n) + i \cdot \text{randn}(n) \]
   \[ S = (A + A^\dagger) / (2\sqrt{n}) \]

   Two facts above remain true.

Notes: p. 48, 135-136 for above.
   p. 63 for below

3. (Chi-distribution and orthogonal invariance)

   Let \( V_n = \text{randn}(n, 1) \), i.e., an \( n \)-vector
   with independent standard normal entries

   a) \( \|V_n\| \) is known as a \( \chi^2 \) variable
      its prob. density is
      \[ \frac{1}{2^{n-1} \Gamma(n/2)} 2^{n-1} e^{-x^2/2} \] on \( \mathbb{R}_+ \)
Obviously \( \chi^2_{n_1} + \chi^2_{n_2} = \chi^2_{n_1+n_2} \)

Degrees of freedom need not be integers, however.

6) Orthogonal Invariance

If \( \Omega \) is fixed or a random and independent of \( V_n \), then

\( \Omega V_n \) and \( \Omega V_n \) are identically distributed.

Joint density = \( \text{C} \cdot \frac{-\frac{3}{2}}{\|V_n\|^2/2} \)

which only depends on \( \|V_n\| \)

\[ V_n = \frac{\|V_n\| \cdot \frac{V_n}{\|V_n\|}} {\|V_n\|} \]

\( \chi_n \) uniform on sphere

\( \chi_n \) independent

4. Central Limit Theorem

Scalar: \( \frac{1}{\sqrt{K}} \sum_{i=1}^{K} X_i \rightarrow \text{Standard Normal} \)

if \( X_i \) have mean 0

variance 1 (+ bounded moments)
Suppose $X_k$ is an $n 	imes n$ matrix with elements of mean 0 and variance 1 (of finite moments)

$$X = \frac{1}{K} \sum_{i=1}^{K} X_k$$

Then $(X + X^T) / \sqrt{n}$ converges to a GUE.

Similarly for a GUE.

Hence, universality already seems likely.

5. Universality is a physics terminology term somewhat related to what appears when considering statistical ensembles. It is an indicator of a central limit theorem effect. If a non-Gaussian random matrix acts approximately, asymptotically, or on average like a Gaussian matrix, we say that universality is present.

6. The "O" in GOE, the "U" in GUE

Let $A = \text{randn}(n)$

Clearly $A, A^2$ is also $\text{randn}(n)$ for orthogonal $A_1, A_2$
So is Q(AQ') rows & columns retained!

Thus (Q(AQ'))' is distributed as A'A

Same with GUE for unitary matrices

7. Theorem: Let $T_n$ be the random sym\ntriangular Matrix

\[
\begin{bmatrix}
\sqrt{a_{11}} & X \beta(n) & X \beta(n-2) & \cdots & X \beta(n-2k+1) \\
\end{bmatrix}
\]

Divide by $\sqrt{a_{11}}$ to get eigenvalues approximately $\varepsilon_i$

When $\beta=1$ the eigenvalues of $T_n$ are distributed\nthe same as the GUE

When $\beta=2$ they are the same as GUE

All $\beta>0$ make sense

Comment: To compute all eigenvalues\nof Gaussian ensembles efficiently, one\ncould use MATLAB's denen eig (calls LAPACK)\nor sparse eig (calls ARPACK) but\nboth are inefficient.
Let $X \sim \text{randn}(n)$

$S = \frac{X + X^T}{\sqrt{2}}$

Diagonal $S \geq 0$, i.e. variance $= 2$

off-diag: variance $= \frac{1}{2} = 1$

Householder reflector from numerical computation

Given $x$ construct $H$ orthogonal so that

$Hx = ||x||e_1$  \hspace{1cm} (e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix})$

\[ \xrightarrow{H} \]

Details don't matter.

One construction $V = X - ||x||e_1 e_1^T$, extend $x$ and $e_1$.

On computers other choices used for stability.

$H$ is thought of as a tool to insert zeros.

Usual Tridiagonal

\[
\begin{pmatrix}
S_2 & S_1 & 0 \\
S_1 & S_2 & 0 \\
\vdots & \vdots & \ddots \\
0 & 0 & 0 & S_{n-1} & S_n
\end{pmatrix}
\]
Brownian Motion + White Noise

\[ h = \frac{1}{n} \]
\[ x = [0, h, 1] \]
\[ dW = \text{randn}(1, 1) \cdot \text{sqrt}(h) \]
\[ W = \text{cumsum}(dW) \]
\[ p(x, x') = \]

Note that \( W(x) \) is normal white noise with variance \( \frac{1}{n} \) and \( x = 1 \)

so \( W(x) \) makes sense as a random variable

\[ dW = (\text{standard normal}) \cdot \sqrt{dx} \]

\[ W(x) = \frac{dW}{dx} \cdot x = \frac{dW}{h} \] discrete time noise process \( W(x) \)
Operators on \((0, \infty)\)

**Discrete**

\[
\begin{bmatrix}
F(0), f(h), F(2h), & \ldots
\end{bmatrix}
\]

**Continuous**

\[
f(x)
\]

1st Derivative: Discrete:

\[
\frac{1}{2}
\]

Continuous:

\[
f(x) \rightarrow \frac{1}{h} f'(x)
\]

2nd Derivative:

\[
\frac{1}{h^2}
\]

Multiplication by \(\alpha\):

\[
\frac{1}{h^2}
\]

Multiplication by \(w(x)\):

\[
\frac{1}{hn}
\]

Let \(T\) be discretization of \(\frac{d^2}{dx^2} - x + \frac{2}{\sqrt{\beta}} w'(x)\)

then \(\text{diag}(T) = \left[ -\frac{2}{h^2} + h \omega + \text{rand}(0)/\text{det}(h); \quad \text{for } i \in N \right] \)

\[\text{sup}(T) = -2/n^2 \times \cong n/(N-1)\]
Easy Theorem: The upper left of the triadical model converges.

\[ \chi_n \approx \sqrt{n} + \frac{c}{\sqrt{n}} \]