Problem 1

(1 – a) --- 15 points

Reducing the block diagram:

\[
G(s) = \frac{s + 1}{s^2 + 1} \frac{s + 2}{s - 2} \]

(1 – b) ----- 15 points

Open Loop Poles and zeros

Poles: \(2s^2 + 5s + 5 = 0\)
\[p_{1,2} = \frac{-5 \pm \sqrt{25 - 40}}{4} = \frac{-5 \pm j\sqrt{15}}{4} \approx -1.25 \pm j1\]
\[p_3 = +2\]

Zeros: \(z_1 = -1, \ z_2 = -2\)

The open loop system is unstable, since there is one pole in the right half plane.
The closed-loop transfer function is given by:

\[ T(s) = \frac{KG}{1 + KG} \]

\[ T(s) = \frac{K(s+1)(s+2)}{(2s^2 + 5s + 5)(s-2) + K(s+1)(s+2)} \]

\[ T(s) = \frac{K(s+1)(s+2)}{2s^3 + (K+1)s^2 + (3K-5)s + (2K-10)} \]

Characteristic equation:

\[ 2s^3 + (K+1)s^2 + (3K-5)s + (2K-10) = 0 \]

The Routh Hurwitz Table:

<table>
<thead>
<tr>
<th>s^3</th>
<th>2</th>
<th>3K-5</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^2</td>
<td>K+1</td>
<td>2K-10</td>
</tr>
<tr>
<td>s^1</td>
<td>b_1</td>
<td>0</td>
</tr>
<tr>
<td>s^0</td>
<td>2K-10</td>
<td>0</td>
</tr>
</tbody>
</table>

Where,

\[ b_1 = \frac{(3K-5)(K+1)-2(2K-10)}{K+1} \]

\[ = \frac{3(K^2 - 2K + 5)}{K+1} \]

For stability all must be greater than zero

- \( K + 1 > 0 \rightarrow K > -1 \)
- \( 2K - 10 > 0 \rightarrow K > 5 \)
- \( K^2 - 2K + 5 > 0 \rightarrow (K-1)^2 + 4 > 0 \)

Therefore this condition is met for all real \( K \).

\[ \therefore \text{ Stability Condition: } K > 5 \]
Problem 2

(2 – a) ----- 15 points

Free-body diagrams

\[ J_1 \ddot{\theta}_1 = \tau - F_i \times 1 \]

\[ 0 = F_i \cdot N - K_1 (\theta_1^* - \theta_2) \]

\[ J_2 \ddot{\theta}_2 = -b \dot{\theta}_2 + K_2 (\theta_1^* - \theta_2) \]

\[ J_1 s^2 N \dot{\theta}_1^* = \tau - \frac{K_1}{N} (\dot{\theta}_1^* - \dot{\theta}_2) \]

\[ J_1 s^2 N \dot{\theta}_1^* = N \tau - K_1 \dot{\theta}_1^* + K_2 \dot{\theta}_2 \]

\[ (J_1 s^2 N^2 + K) \dot{\theta}_1^* = N \tau - K_2 \dot{\theta}_2 \]

\[ \dot{\theta}_1^* = \frac{N \tau + K_2 \dot{\theta}_2}{J_1 s^2 N^2 + K} \]

\[ (J_2 s^2 + bs + K) \theta_2 = K_1 \theta_1^* \]

and output \( y \) is given by

\[ y = L \theta_2 \]

Solving for \( y \) yields:

\[ G(s) = \frac{y(s)}{\tau(s)} = \frac{K_i LN}{s [N^2 J_1 J_2 s^3 + N^2 J_1 b s^2 + K_i (N^2 J_1 + J_2) s + K_i b]} \]

(2 – b) ----- 20 points

Obtaining the closed-loop transfer function,

\[ T(s) = \frac{k_p G}{1 + k_p G} = \frac{5k_p}{s(s^3 + 12s^2 + 66s + 132) + 5k_p} \]

The characteristic equation is:
\[ C.E. = s^4 + 12s^3 + 66s^2 + 132s + 5k_p = 0 \]

Using the Routh table:

<table>
<thead>
<tr>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
<th>( s^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12 → 1</td>
<td>12 → 11</td>
<td>11 → 0</td>
<td>0</td>
</tr>
<tr>
<td>66</td>
<td>55 → 11</td>
<td>5k_p → k_p</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5k_p</td>
<td>121−k_p</td>
<td>0</td>
<td>11</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>k_p</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

For positive \( k_p \) the elements of the first column in the Routh table are all positive except for \( s^1 \). The closed-loop system becomes marginally stable when this coefficient becomes zero:

\[ \frac{121−k_p}{11} = 0, \quad \therefore k_p = 121 \]

When \( k_p = 121 \) the row \( s^1 \) becomes all zero; this will make the coefficients of \( s^2 \) be a factor of the characteristic equation.

\[ 11s^2 + k_p = 0 \]

\[ s = \pm j \sqrt{\frac{k_p}{11}} = \pm j \sqrt{11} \]

The oscillatory frequency is:

\[ \omega_n = \sqrt{11} \text{ rad/s} \]

(2 - c) ----- 15 points

The poles closest to the imaginary axis \( p_{1,2} \) dominate the response

\[ p_{1,2} = -1 \pm j2 \]

\[ G(s) = \frac{y(s)}{\tau(s)} = \frac{K_i LN}{s[N^2J_1J_2s^3 + N^2J_1bs^2 + K_i(N^2J_1 + J_2)s + K_i b]} \]

\[ \omega_d = 2, \quad \sigma = 1 \]

\[ (s + 1 - j2)(s + 1 + j2) = s^2 + 2s + 5 \]

Settling time:

\[ T_s = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma} = 4 \text{ sec} \]