Problem 1: There is something wrong somewhere! In Quiz 2, Problem 1 we were given a system in diagonal form, similar to
\[
\begin{align*}
\dot{x} &= \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} x
\end{align*}
\]
and asked to investigate its controllability. Let’s take that a bit further – here are some general statements that do not seem to be consistent:

1. Controllability is a property of a system that is invariant under a similarity transform \( X = Pz \) so that
\[
\dot{z} = P^{-1}APz + P^{-1}Bu
\]

2. The above system has controllability matrix
\[
\Theta_c = \begin{bmatrix} 1 & \alpha_1 \\ 0 & \alpha_2 \end{bmatrix}
\]
which indicates that the system is uncontrollable when the eigenvalues are equal \((\alpha_1 = \alpha_2)\), which we would expect.

3. The above system has a transfer function
\[
H(s) = C [sI - A]^{-1} B = \frac{(c_1 + c_2)s - (\alpha_1 c_2 + \alpha_2 c_1)}{s^2 - (\alpha_1 + \alpha_2)s + \alpha_1 \alpha_2}
\]

4. The transfer function allows us to write the system in control canonical (phase-variable) form by inspection
\[
\begin{align*}
\dot{z} &= \begin{bmatrix} 0 & 1 \\ -\alpha_1 \alpha_2 & \alpha_1 + \alpha_2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} -(\alpha_1 c_2 + \alpha_2 c_1) & c_1 + c_2 \end{bmatrix} z
\end{align*}
\]
(we could have found the same result using \( \Theta_c \) in the matrix \( P \).

5. The transformed system has controllability matrix
\[
\Theta_c = \begin{bmatrix} 0 & 1 \\ 1 & \alpha_1 + \alpha_2 \end{bmatrix}
\]
which indicates that the system is always controllable (including when \( \alpha_1 = \alpha_2 \)).
This argument seems to indicate that the original system is uncontrollable when the eigenvalues are coincident but the equivalent transformed system does not. The question is, how do we reconcile Statement 1 with the results implied by Statements 2 and 5? Your task is to find the fallacy in the argument.

**Problem 2:** A system is represented by

\[ A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \]

(a) Find a constant state feedback matrix \( K \) which yields closed-loop poles \( \lambda_{1,2,3} = -2, -3, -4 \).

(b) Find a constant state feedback matrix \( K \) which yields closed-loop poles \( \lambda_{1,2,3} = -3, -2 + j, -2 - j \).

**Problem 3:** A simple robot arm, moving in a horizontal plane, is controlled by an electric motor \( T_m \) through a compliant belt with stiffness \( k \). The arm is represented by an inertia \( J_2 \), and is subject to an external disturbance torque \( T_d \).

The state equations are given by

\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-k/J_1 & -b_1/J_1 & k/J_1 & 0 \\
0 & 0 & -k/J_2 & -b_2/J_2 \\
k/J_2 & 0 & 0 & -k/J_2
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\dot{\theta}_1 \\
\theta_2 \\
\dot{\theta}_2
\end{bmatrix} +
\begin{bmatrix}
0 & 0 \\
1/J_1 & 0 \\
0 & 0 \\
0 & 1/J_2
\end{bmatrix}
\begin{bmatrix}
T_m \\
T_d
\end{bmatrix}
\]

where \( \theta_1 \) and \( \theta_2 \) are the angles of the motor shaft and robot arm respectively. Assume the following numerical values: \( J_1 = 10, J_2 = 5, k = 160, b_1 = 20, b_2 = 5 \).

(a) Find transfer functions relating the arm angle \( \theta_2 \) to the torques \( T_m \) and \( T_d \).

(b) Assume both \( T_m \), and \( T_d \) as inputs, find a state feedback matrix \( K \) that will place the closed-loop eigenvalues on a “Butterworth” semi-circle of radius 2 in the left-half plane, as shown below.
Problem 4: The plant in a dc motor speed control system is described by the model

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w \\
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x
\]

where \( x_1 \) is the motor speed, \( x_2 \) is the armature current, \( u \) is the applied voltage and \( w \) is the load torque. Design a state-feedback plus integral control so that the closed-loop poles will be located at \(-1, -1\) and \(-2\). (Assume that the load torque is a disturbance, and that the controlled input is motor voltage.) Make plots of the response of the system to a step change in the commanded speed, and to a step change in the load torque.