Soon, we’re going to take our first serious look into the intricacies of electronic hardware. Before we do, there are two concepts that it pays to be clear on. 1.) Frequency domain representations of signals; and 2.) voltage dividers. The first involves knowing a little bit about the famous Fourier transform. The second is a tremendously useful result that you can derive from the node equations. So let’s get started…

…with the simple RC circuit shown below.

Suppose that the voltage source is “off” and we turn it on a time $t=0$. You all have the analytical tools to dive right into this, but I want to take the unconventional approach at first. The point is to show you where differential equations come from. Once we are confident and sure that differential equations are to be trusted, the leap to frequency domain thinking is not a giant one.
Start with what we know:

Ohms Law: \[ V = IR \Rightarrow I = \frac{V_0 - V_C}{R} \]

Capacitor Definition: \( Q = CV \Rightarrow V_C = \frac{Q}{C} \)

Now you’re used to crawling right through from here, but pause to consider how awkward our situation is. If \( V_C = 0 \) to start, we have \( \frac{V_0}{R} \) Coulombs/sec of charge flowing to the capacitor, \( |V_0 - V_C| \) gets smaller and so the current decreases…it gives you a headache trying to sort it all out.

We might hope that we could get out of this little trap by discretizing time, and just make sure that our increments in time were suitably “small.” How would that bit of reasoning play out in the language of math?

\[
V_C(t + \Delta t) = V_C(t) + \frac{V_0 - V_C(t)}{R} \Delta t \cdot \frac{1}{C} \\
\Delta I \quad \Delta Q \\
\frac{\Delta Q}{C} = \Delta V
\]

Rearrange: \[
\frac{V_C(t + \Delta t) - V_C(t)}{\Delta t} + \frac{1}{RC} V_C(t) = \frac{1}{RC} V_0(t)
\]
Now what about making it “suitably small”?

\[
\lim_{\Delta t \to 0} \left[ \frac{V_C(t + \Delta t) - V_C(t)}{\Delta t} \right] + \frac{1}{RC} V_C(t) = \frac{1}{RC} V_0(t)
\]

We’ve seen this before!

\[
RC \frac{dV_C(t)}{dt} + V_C(t) = V_0(t)
\]

So…in this case, a differential equation is a perfectly natural way to describe the time evolution of this system. We didn’t make up notation just to torture college students…

We just got tired of writing \( \lim_{\Delta t \to 0} \) all the time. Seriously.

This seems like a trivial example. But it turns out that with some clever manipulations, you can extend the same thinking to higher order differential equations.

Why are we so happy about this? Because a fantastic fact about systems covered by linear differential equations with constant coefficients is that they behave so beautifully when driven by sinusoids. Suppose, for instance, we made \( V_0(t) \) a sinusoid, \( e^{j\omega_0 t} \).

Recall \( e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t \)

[Whoa! Imaginary numbers to describe real signals?!? More on that in a minute.]
If \( V_0(t) = e^{j\omega_0 t} \), we “guess” \( V_C = Ae^{j\omega_0 t} \), and solve for \( A \):

\[
RCj\omega_0 Ae^{j\omega_0 t} + Ae^{j\omega_0 t} = e^{j\omega_0 t}
\]

\[
A = \frac{1}{j\omega_0 RC + 1}
\]

So, \( V_C = \frac{1}{j\omega_0 RC + 1} e^{j\omega_0 t} \)

Class Exercise:

Show that if \( V_0 = e^{j\omega_0 t} + e^{j\omega_1 t} \), the solution for \( V_C(t) \) is

\[
V_C(t) = \frac{1}{j\omega_0 RC + 1} e^{j\omega_0 t} + \frac{1}{j\omega_1 RC + 1} e^{j\omega_1 t}
\]

(Workspace)

That, friends, is the great thing about linearity.
Now, about this imaginary number funny business. In the real world, you use real sinusoids like

\[ V_0(t) = \cos \omega_0 t = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} \]

\[ V_c(t) = \frac{1/2}{j\omega_0 RC + 1} e^{j\omega_0 t} + \frac{1/2}{-j\omega_0 RC + 1} e^{-j\omega_0 t} \]

Complex conjugates

…and when you sum complex conjugates, the imaginary part vanishes.

\[ x = a + jb \quad x^* = a - jb \]

\[ x + x^* = 2a \]

If we put real signals in, we get real results out. I haven’t done a formal proof here, but if you need one it is not hard to do.

We can go further. Suppose I imagine a signal that is composed of a large number of sinusoids:

\[ y(t) = \sum_k A_k e^{j(k\Delta \omega)t} \]

(If I’m concerned about keeping y(t) real, I require \( A_k = A_{-k}^* \).)
Following our script, we have for the capacitor voltage

\[ V_C(t) = \sum_k \frac{A_k}{j(k \cdot \Delta \omega)RC + 1} e^{i(k \cdot \Delta \omega)t} \]

Interesting! What do we see for this humble circuit?

- For low frequencies \( (k \cdot \Delta \omega)RC \ll 1 \), this circuit passes sinusoids through unmolested.
- For high frequencies, \( (k \cdot \Delta \omega)RC \gg 1 \), this circuit attenuates sinusoids heavily. (And imparts a phase shift of \( 90^\circ \).)

Plotting this “frequency response”

A low pass filter!

This is all great, but there is one vital piece missing. This analysis only seems to work if our input happens to be a sum of sinusoids. But how common is that?

Answer: Extremely common. This is what the Fourier transform is all about. The Fourier transform establishes the following equivalence:
\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t} d\omega \]

\[ Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t} dt \]

But that doesn’t look like a sum of sinusoids, does it? Remember what an integral is:

\[ y(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k} Y(jk\Delta\omega)e^{j(k\omega)\Delta\omega} \Delta\omega \]

We just get tired of writing \( \lim_{\Delta\omega \to 0} \) over and over again. So we give ourselves a fancy new integral sign and call it a day. Note that linearity lets us write \( V_C(j\omega) = \frac{1}{j\omega + 1} Y(j\omega) \), and then we have

\[ V_C(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{j\omega + 1} Y(j\omega)e^{j\omega t} d\omega. \]

This way of thinking is practically in the DNA of every electrical engineer. In our minds, we see all signals as broken down into their constituent sinusoids, and we speak freely about what a network “does” to high frequency components and low-frequency components.

Now, voltage dividers. What do they mean?
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Lecture 2: Frequency Domain Thinking
Prof. Joel L. Dawson

\[ Z_1 \text{ and } Z_2 \text{ are “impedances,” and their definition takes for granted that we have split our input into a bunch of sinusoids using the Fourier transform. Recall Ohm’s Law; } V = IR. \text{ Well, with impedance we say that } V = IZ, \text{ where:} \]

**Resistor**
\[
\begin{align*}
\mathcal{R} \hspace{1cm} & +V - \\
\implies & V = IZ \\
\end{align*}
\]
\[ Z = \frac{V}{I} = R \]

**Capacitor**
\[
\begin{align*}
\mathcal{C} \hspace{1cm} & +V_i - \\
\implies & I = C \frac{dV}{dt} \rightarrow I(j\omega) = j\omega CV_i(j\omega) \\
\end{align*}
\]
\[ Z = \frac{V(j\omega)}{I(j\omega)} = \frac{1}{j\omega C} \]

**Inductor**
\[
\begin{align*}
\mathcal{L} \hspace{1cm} & +V_i - \\
\implies & V_i = L \frac{dI}{dt} \\
\end{align*}
\]
\[ Z = \frac{V(j\omega)}{I(j\omega)} = j\omega L \]

Using the concepts of the voltage divider, impedance, and frequency domain thinking, you can get a lot of understanding out of quick looks at networks: