Exercise 6-1: Consider an amplifier with an input-output relation that takes the form \( v_{\text{OUT}} = V_A (v_{\text{IN}}/V_B)^3 \), where \( V_A \) and \( V_B \) are constants. Determine its input bias voltage \( V_{\text{IN}} \) and its small-signal gain \( v_{\text{OUT}}/v_{\text{IN}} \) for a given output bias voltage \( V_{\text{OUT}} \).

Answer: For a given output bias voltage \( V_{\text{OUT}} \), we can readily compute the corresponding input bias voltage \( V_{\text{IN}} \) as

\[
V_{\text{IN}} = V_B \left( \frac{v_{\text{OUT}}}{V_A} \right)^{\frac{1}{3}} \bigg|_{v_{\text{OUT}}=V_{\text{OUT}}} 
\]

The small signal gain is given by

\[
v_{\text{out}} = \frac{3V_A}{V_B^3} \left( \frac{v_{\text{IN}}}{V_A} \right)^{\frac{2}{3}} v_{\text{IN}} = \frac{3V_A}{V_B^3} \left( \frac{V_{\text{OUT}}}{V_A} \right)^{\frac{2}{3}} v_{\text{IN}} = \frac{3V_A}{V_B^3} \left( \frac{V_{\text{OUT}}}{V_A} \right)^{\frac{2}{3}} v_{\text{IN}} = \frac{V_A}{V_B} \frac{V_{\text{OUT}}}{V_A} v_{\text{IN}},
\]

thus \( \text{gain} = 3 \frac{V_A}{V_B} \frac{V_{\text{OUT}}}{v_{\text{IN}}} \).

Exercise 6-2: Find the capacitance of the all-capacitor network, and the inductance of the all-inductor network, shown below.
Answer:

\[
C_{eq} = \left( \frac{1}{C_1} + \frac{1}{C_2 + C_3} \right)^{-1} = \frac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3}
\]

\[
L_{eq} = \left( \frac{1}{L_1} + \frac{1}{L_2 + L_3} \right)^{-1} = \frac{L_1(L_2 + L_3)}{L_1 + L_2 + L_3}
\]

Exercise 6-3: Each network shown below has a single capacitor, and hence a single resistor-capacitor time constant. Find the time constant for each network. Hint: consider the Thevenin equivalent network that drives each capacitor.

Answer: Looking in from the capacitor ports of each networks as shown below

we compute the Thevenin equivalent resistances of respective networks as

\[
R_{TH,1} = (R_1 + R_2) \parallel R_3 \\
R_{TH,2} = R_1 \parallel R_2 + R_3
\]

Thus, the time constants are given by

\[
\tau_1 = R_{TH,1} C \\
\tau_2 = R_{TH,2} C
\]

Problem 6-1: The network shown below is driven by a voltage step at \( t = 0 \). The current through the inductor at \( t = 0^- \) is \( i_0 \). Find the current \( i(t) \) through the inductor for \( t \geq 0 \) as follows.
(a) Use appropriate Thevenin and/or Norton transformations to simplify the network, and then use nodal analysis to find a differential equation for \( i(t) \) which is driven by \( V(t) \); analysis by the node method alone will be too much work. Note that the initial condition for this differential equation is \( i(0^-) = i_0 \).

![Circuit Diagram]

**Answer:** Computing Thevenin voltage and resistance, consider the following equivalent network, where

\[
V_{TH}(t) = V(t) \frac{R_2}{R_1 + R_2}
\]  

(7)

and

\[
R_{TH} = R_3 + R_1 \parallel R_2
\]  

(8)

We can then write the appropriate differential equation for the network by applying KVL around the loop in the Thevenin circuit as

\[
\frac{L}{R_{TH}} \frac{d}{dt} i(t) + i(t) = \frac{V_{TH}(t)}{R_{TH}}
\]  

(9)

(b) Find \( i(0^+) \) by integrating the differential equation from \( t = 0^- \) to \( t = 0^+ \) under the assumption that \( i(t) \) is finite during that time. Note that \( V(t) \) is always finite by definition. Is your answer self consistent with the assumption? **Answer:** Integrating Eq. (9) from \( t = 0^- \) to \( t = 0^+ \) yields

\[
\frac{L}{R_{th}} \int_{i(0^-)}^{i(0^+)} di + \int_{t=0^-}^{t=0^+} i dt = \frac{1}{R_{th}} \int_{t=0^-}^{t=0^+} V_{th}(t) dt
\]  

(10)
For finite \( i(t) \) and \( V_{TH}(t) \)

\[
\int_{t=0-}^{t=0+} i(t) \, dt = 0 \quad \text{and} \quad \int_{t=0-}^{t=0+} V_{TH}(t) \, dt = 0
\]  

(11)

Thus, Eq. (10) reduces to

\[
i(t = 0^+) = i(t = 0^-)
\]  

(12)

This result is self consistent with the assumption that the current through inductor does not change instantaneously.

(c) Beginning at \( t = 0^+ \), find a particular solution \( i_p(t) \) to the differential equation. The particular solution must satisfy the driven differential equation, but it need not match the initial condition.

**Answer:** Let \( i_p(t) \) be the particular solution to Eq. (9) for \( t \geq 0^+ \) given \( V_{TH} = R_2/(R_1 + R_2) \). Solving the equation yields

\[
i_p(t) = \frac{V_{TH}(t)}{R_{TH}}
\]  

(13)

(d) Find the homogeneous solution \( i_h(t) \) to the undriven differential equation. Note that the homogeneous equation must satisfy \( i_h(0^+) = i(0^+) - i_p(0^+) \).

**Answer:** Let \( i_h(t) \) be the homogeneous solution to Eq. (9) for \( t \geq 0^+ \), i.e.

\[
\frac{L}{R_{TH}} \frac{d}{dt} i_h(t) + i_h(t) = 0
\]  

(14)

Solving \( i(t) \) from Eq. (14) yields

\[
i_h(t) = A e^{-\frac{t}{L/R_{TH}}}
\]  

(15)

Since \( i_h(0^+) = A \) must satisfy \( i_h(0^+) = i(0^+) - i_p(0^+) \) we can deduce that \( A = i_0 - V_{TH}/R_{TH} \) or

\[
i_h(t) = (i_0 - \frac{V_{TH}}{R_{TH}}) e^{-\frac{t}{(L/R_{TH})}}
\]  

(16)

(e) Find the complete solution to the problem by adding the particular and homogeneous solutions: \( i(t) = i_p(t) + i_h(t) \) for \( t \geq 0^+ \).

**Answer:** It is now only the formality.

\[
i(t) = \frac{V_{TH}}{R_{TH}} + \left( i_0 - \frac{V_{TH}}{R_{TH}} \right) e^{-\frac{t}{(L/R_{TH})}}
\]  

\[
= \frac{V_{TH}}{R_{TH}} \left( 1 - e^{-\frac{t}{(L/R_{TH})}} \right) + i_0 e^{-\frac{t}{(L/R_{TH})}}
\]  

(17)

**Problem 6-2:** The network shown below has two ports, two resistors and one capacitor. The resistor values \( R_1 \) and \( R_2 \), and the capacitor value \( C \), are unknown. Also shown below is the result of an experiment performed on the network in which one port is driven with the voltage step \( v_{IN} \) at \( t = 0 \), and the other port is loaded with a 1 k\( \Omega \) resistor. Using the experimental result, which consists of the measured current \( i_{IN} \), find the values \( R_1 \), \( R_2 \) and \( C \). Also, find the voltage \( v_{OUT} \).
across the load resistor for $t \geq 0$. In doing so, assume that the network capacitor is uncharged prior to $t = 0$.

Since the network capacitor is uncharged prior to $t = 0$, $v_C(0^-) = v_C(0^+) = 0$.

At $t = 0$ the capacitor looks like short circuit. Thus, the circuit reduces to $v_{in}(t)$ in series with $R_1$, where $v_{in} = 6\,[V]$ and $i_{in} = 2\,[mA]$. Therefore, $R_1 = 3\,[k\Omega]$.

As $t \to \infty$ the capacitor looks like open circuit. Thus, the circuit reduces to $v_{in}(t)$ in series with $R_1$, $R_2$, and the 1$K\Omega$ resistor, where $v_{in} = 6\,[V]$ and $i_{in} = 1\,[mA]$. Therefore, $R_2 = 2\,[k\Omega]$.

Looking in from the capacitor port the Thevenin resistance is given by

$$R_{TH} = R_1 \parallel (R_2 + R_3), \quad \text{where} \quad R_3 = 1K\Omega$$

$$= \frac{3}{2}\,[k\Omega]$$

(18)

As in Exercise 6-3 we can relate the time constant with the product of capacitance and Thevenin resistance of the circuit, i.e.

$$\tau = 0.5 \,[ms]$$

$$= R_{TH} C$$

$$\Rightarrow \quad C = \frac{1}{3}\,[\mu F]$$

Finally, $v_{OUT}$ is given as

$$v_{OUT} = \frac{1}{3}v_C$$

(19)

where,

$$v_c(t) = V_{in}(t) - R_1 i_{in}(t)$$

$$= 3\left(1 - e^{-\frac{t}{0.5\,\text{ms}}})\right)$$

(20)
Problem 6-3: At $t = 0^-$, the networks shown below have zero initial state. That is, the capacitor voltage $v(t)$ and the inductor current $i(t)$ are both zero at $t = 0^-$. At $t = 0$, the current source produces an impulse of area $Q$, and the voltage source produces an impulse of area $\Lambda$.

\[ \text{I(t)} \quad R_2 \quad R_1 \quad C \quad v(t) \quad \text{V(t)} \quad R_1 \quad R_2 \quad L \quad i(t) \]

\[ \text{I(t)} \quad Q \quad t \quad 0 \]

\[ \text{V(t)} \quad \Lambda \quad t \quad 0 \]

(a) Find the capacitor voltage $v(t)$ and the inductor current $i(t)$ at both $t = 0^+$ and $t = \infty$, and explain the results physically. One way to find the states at $t = 0^+$ is to first write the appropriate differential equations for the networks, and integrate them from $t = 0^-$ to $t = 0^+$ under the assumption that the states remain finite during that time; you should justify this assumption. Then, substitute the initial conditions at $t = 0^-$ into the results to determine the states at $t = 0^+$. 

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Answer: First we reduced the circuit to their Norton and Thevenin equivalents respectively as shown below where

\[ R_N = R_1 + R_2 \]
\[ I_N(t) = \frac{R_1}{R_1 + R_2} I(t) \] \hspace{1cm} (21)

for the capacitor circuit and

\[ R_{TH} = R_1 \parallel R_2 \]
\[ V_{TH}(t) = \frac{R_2}{R_1 + R_2} V(t) \] \hspace{1cm} (22)

for the inductor one.

By applying KCL to the capacitor circuit we obtain the following differential equation:

\[ \frac{1}{R_N} v(t) + C \frac{dv(t)}{dt} - I_N(t) = 0 \quad \text{or} \]
\[ \frac{dv(t)}{dt} + \frac{1}{R_N C} v(t) = \frac{1}{C} I_N(t) \] \hspace{1cm} (23)

Integrating both sides from \( t = 0^- \) to \( t = 0^+ \) under the assumption that \( v(t) \) is finite during that time we obtain

\[ \int_{v(0^-)}^{v(0^+)} dv + \frac{1}{R_N C} \int_{t=0^-}^{t=0^+} v(t) dt = \frac{1}{C} \int_{t=0^-}^{t=0^+} I_N(t) dt \]
\[ v(0^+) - v(0^-) = \frac{1}{C} \left( \frac{R_1}{R_1 + R_2} \right) Q \]
\[ v(0^+) = \left( \frac{R_1}{R_1 + R_2} \right) \frac{Q}{C} \quad \text{[Volts]} \] \hspace{1cm} (24)

Note that \( Q \) has the unit of coulombs.

As \( t \to \infty \) the capacitor looks like open circuit with \( I_N = 0 \). Thus, the voltage across \( R_N \) becomes zero or

\[ v(\infty) = 0 \] \hspace{1cm} (25)

Similarly by applying KVL to the inductor circuit we obtain the following differential equation:

\[ R_{TH} i(t) + L \frac{di(t)}{dt} - V_{TH}(t) = 0 \quad \text{or} \]
\[ \frac{di(t)}{dt} + \frac{R_{TH}}{L} i(t) = \frac{1}{L} V_{TH}(t) \] \hspace{1cm} (26)
Integrating both sides from \( t = 0^- \) to \( t = 0^+ \) under the assumption that \( i(t) \) is finite during that time we obtain

\[
\int_{t=0^-}^{t=0^+} di + \frac{R_{TH}}{L} \int_{t=0^-}^{t=0^+} i(t)dt = \frac{1}{L} \int_{t=0^-}^{t=0^+} V_{TH}(t)dt
\]

\[
i(0^+) - i(0^-) = \frac{1}{L} \left( \frac{R_2}{R_1 + R_2} \right) \Lambda
\]

\[
i(0^+) = \left( \frac{R_2}{R_1 + R_2} \right) \frac{\Lambda}{L} \text{ [Amps]}
\]

Note that \( \Lambda \) has the unit of webers in this case.

As \( t \to \infty \) the inductor looks like short circuit with \( V_{TH} = 0 \). Thus, the current through \( R_{TH} \) becomes zero or

\[
i(\infty) = 0
\]

(b) Next, find the time constant by which the capacitor voltage or inductor current goes from its initial value at \( t = 0^+ \) to its final value at \( t = \infty \).

\textbf{Answer:} The time constants can be readily deduced as

\[
\tau_{cap} = R_N C
\]

and

\[
\tau_{ind} = \frac{L}{R_{TH}}
\]

from the Norton and Thevenin equivalent circuits respectively.

(c) Using the previous results, and without necessarily solving the differential equation directly, construct \( v(t) \) and \( i(t) \) for \( t \geq 0 \).

\textbf{Answer:} Putting the answers obtained in parts (a) and (b) together we construct \( v(t) \) and \( i(t) \) for \( t \geq 0 \) as

\[
v(t) = \left( \frac{R_1}{R_1 + R_2} \right) \frac{Q}{C} e^{-\frac{t}{\tau_{cap}}} \text{ [Volts]}
\]

\[
i(t) = \left( \frac{R_2}{R_1 + R_2} \right) \frac{\Lambda}{L} e^{-\frac{t}{\tau_{ind}}} \text{ [Amps]}
\]

\textbf{Problem 6-4:} This problem examines the relation between transient responses of linear systems. The network shown below is first driven by a step at \( t = 0 \), and then next driven by a ramp at \( t = 0 \). In both cases, the inductor has zero initial current.

\textbf{Answer:} The differential equation for the circuit can be written as

\[
\frac{L di}{R dt} + i(t) = I(t)
\]

(a) Find the inductor current \( i(t) \) in response to the step shown below.

\textbf{Answer:} Solving Eq. (32) for particular solution and homogenous one given step input we obtain

\[
i_p(t) = I_0
\]

\[
i_H(t) = -I_0 e^{-\frac{t}{\tau_{ind}}}
\]
Therefore, the current response to the step is given by

\[ i(t) = I_o \left( 1 - e^{-\frac{t}{\alpha L/R}} \right) \]  

(b) Find the inductor current \( i(t) \) in response to the ramp shown below.

**Answer:**  Guess the particular solution to Eq. (32) to be

\[ i_p(t) = At + B \]  

Then Eq. (32) reduces to,

\[ \frac{L}{R} A + At + B = \alpha I_o t \]  

Thus,

\[ A = \alpha I_o \]
\[ B = -\frac{L}{R} \alpha I_o \]  

Solving for the homogenous solution to match the initial condition leads to

\[ i_H(t) = \frac{L}{R} \alpha I_o e^{-\frac{t}{\alpha L/R}} \]  

Therefore, the current response to the ramp is given by

\[ i(t) = \alpha I_o t - \frac{L}{R} \alpha I_o \left( 1 - e^{-\frac{t}{\alpha L/R}} \right) \]  

(c) The step input can be constructed from the ramp input according to \( I_{\text{Step}}(t) = \frac{1}{\alpha} \frac{d}{dt} I_{\text{Ramp}}(t) \).

**Show** that their respective responses are related in a similar manner.

**Answer:**  From Eq. (39)

\[ \frac{1}{\alpha} \frac{d}{dt} I_{\text{Ramp}}(t) = \frac{1}{\alpha} \frac{d}{dt} \left( \alpha I_o t - L \alpha I_o \left( 1 - e^{-\frac{t}{\alpha L/R}} \right) \right) \]
\[ = I_o - I_o e^{-\frac{t}{\alpha L/R}} \]
\[ = I_o \left( 1 - e^{-\frac{t}{\alpha L/R}} \right) = I_{\text{Step}}(t) \]  

(d) Would the result from Part C hold if \( i(0^-) \neq 0 \)? Why or why not?

**Answer:**  The result does not hold if \( i(0^-) \neq 0 \). When we differentiate the input, we only differentiate the external source, since the initial condition is internal source. However, when we differentiate the output, we differentiate both the homogeneous and the particular solutions. Mathematically, the term corresponding to initial condition of \( I_{\text{Ramp}}(t) \) gets multiplied by the time constant when differentiated to yield \( I_{\text{Step}}(t) \).