Exercise 7-1: Using one 1mH inductor and two resistors, construct a two-port network that has the following zero-state response to a 1-V step input as shown below. Provide a diagram of the network, and specify the values of the two resistors.

Answer: A moment of thought tells us that the only way to assemble two resistors and an inductor to yield the given zero-state response to a 1-V step input is as shown below.

\[
\begin{align*}
\text{At } t = 0 \text{ the inductor looks like an open circuit. Thus, we have a simple voltage divider where } & R_1 = R_2 = R \text{ to produce } v_2(t = 0) = 0.5 \text{V. For the given time constant, } \\
\frac{L}{R_{TH}} & = \frac{1 \times 10^{-3}}{R_1 \parallel R_2} \\
& = \frac{1 \times 10^{-3}}{R/2}, \text{ since } (R_1 \parallel R_2) = \frac{R}{2} \\
& = 2 \times 10^{-3} \\
\implies R_1 = R_2 = R & = 1\Omega
\end{align*}
\]

Exercise 7-2: All networks shown below begin operation at \( t = 0^- \) with zero capacitor voltage or zero inductor current. That is, all states are zero at \( t = 0^- \). For each network, find the network state, that is the capacitor voltage or inductor current, at both \( t = 0^+ \) and \( t = \infty \). Also find the time constant by which the network state goes from its initial value at \( t = 0^+ \) to its final value at \( t = \infty \). Finally, without actually solving an appropriate differential equation, find the network state for each network for \( 0^+ \leq t \leq \infty \).
Answer: For the first circuit

\[ i_L(t = 0^+) = Q \frac{R}{L} \]
\[ i_L(t = \infty) = 0 \]
\[ \tau = \frac{L}{R} \]
\[ \implies i_L(0^+ \leq t \leq \infty) = Q \left( \frac{R}{L} \right) e^{-\frac{t}{\tau}} \]  

For the second circuit

\[ i_L(t = 0^+) = 0 \]
\[ i_L(t = \infty) = \frac{V_0}{R} \]
\[ \tau = \frac{L}{R} \]
\[ \implies i_L(0^+ \leq t \leq \infty) = \frac{V_0}{R} \left( 1 - e^{-\frac{t}{\tau}} \right) \]  

For the third circuit

\[ v_C(t = 0^+) = 0 \]
\[ v_C(t = \infty) = I_0 R \]
\[ \tau = RC \]
\[ \implies v_C(0^+ \leq t \leq \infty) = I_0 R \left( 1 - e^{-\frac{t}{\tau}} \right) \]  

For the fourth circuit

\[ v_C(t = 0^+) = \frac{\Delta_0}{RC} \]
\[ v_C(t = \infty) = 0 \]
\[ \tau = RC \]
\[ \implies v_C(0^+ \leq t \leq \infty) = \frac{\Delta_0}{RC} e^{-\frac{t}{\tau}} \]
Problem 7-1: This problem explores the basis for treating an initial condition as a network input. Doing so allows the network response to the initial condition to be superimposed on the response to other inputs. Thus, this problem explores the basis behind creating a network response through the superposition of a ZIR and a ZSR.

(a) Network #1 is simply a capacitor with the initial condition of $v_0$ at $t = 0$. For this network, find $v(t)$ as a function of $i(t)$ for $t \geq 0$.

Answer: Employing the concept of capacitor being a device with a memory

$$C \frac{dv_C(t)}{dt} = i_C(t)$$

$$\Rightarrow v_C(t) = \frac{1}{C} \int_{-\infty}^{t} i_C(\tau)d\tau = \frac{1}{C} \int_{0}^{t} i_C(\tau)d\tau + v_C(t = 0)$$

$$v(t) = \frac{1}{C} \int_{0}^{t} i(\tau)d\tau + v_0 \quad \text{since} \quad v_C(t) = v(t) \quad \text{and} \quad i_C(t) = i(t) \quad (6)$$

(b) The capacitor in Network #2 has zero initial condition at $t = 0$. For this network, find $v(t)$ as a function of $i(t)$ for $t \geq 0$.

Answer: Similarly,

$$v_C(t) = \frac{1}{C} \int_{0}^{t} i_C(\tau)d\tau + v_C(t = 0)$$

$$= \frac{1}{C} \int_{0}^{t} i(\tau)d\tau$$

$$\Rightarrow v(t) = \frac{1}{C} \int_{0}^{t} i(\tau)d\tau + V_0 \quad \text{since} \quad v(t) = v_C(t) + V_0 \quad \text{and} \quad i(t) = i_C(t) \quad (7)$$

(c) What must be the relation between $v_0$ in Network #1 and $V_0$ in Network #2 for the terminal relations between $v(t)$ and $i(t)$ to be identical, thereby permitting one network to substitute for the other. With this substitution, the initial capacitor voltage may be treated as a source, thereby allowing superposition.

Answer: In order for Eq. (6) and Eq. (7) to be equal to each other,

$$v_0 = V_0 \quad (8)$$
(d) Network #3 is simply an inductor with the initial condition of \(i_o\) at \(t = 0\). For this network, find \(i(t)\) as a function of \(v(t)\) for \(t \geq 0\).

**Answer:** Analogously employing the idea of inductor being a device with a memory

\[
L \frac{di_L(t)}{dt} = v_L(t) \quad \Rightarrow \quad i(t) = \frac{1}{L} \int_0^t v(\tau)d\tau + i_0
\] (9)

(e) Construct a network containing and inductor and a current source which has the same relation between \(i(t)\) and \(v(t)\) for \(t \geq 0\) as does Network #3. The inductor in the new network must have a zero initial condition.

**Answer:** The equivalent network is given below.

![Network Diagram](image)

**Problem 7-2:** This problem is a continuation of Problem 6.1, and explores a slightly different method of solution. You should think about which method you find easiest.

(a) Find the zero-input response for Problem 6.1. That is, find the response to \(i_o\) with \(V_o = 0\).

**Answer:** The zero-input response is given by

\[
i(t) = i_o e^{-\frac{t}{\tau}}, \quad \text{where} \quad \tau = \frac{L}{R_{TH}}, \quad \text{and} \quad R_{TH} = (R_1 \parallel R_2) + R_3
\] (10)

(b) Find the zero-state response for Problem 6.1. That is, find the response to \(V_o\) with \(i_o = 0\).

**Answer:** The zero-state response is given by

\[
i(t) = \frac{V_{TH}}{R_{TH}} \left(1 - e^{-\frac{t}{\tau}}\right), \quad \text{where} \quad V_{TH} = \frac{R_2}{R_1 + R_2} V(t)
\] (11)

(c) Using superposition, add the two responses to obtain the total response. This should yield the same response you found in Problem 6.1.

**Answer:**

\[
i(t) = \frac{V_{TH}}{R_{TH}} \left(1 - e^{-\frac{t}{\tau}}\right) + i_o e^{-\frac{t}{\tau}}
\] (12)
Problem 7-3: This problem examines the power dissipated by a small digital logic circuit. The circuit comprises a series-connected inverter and NOR gate as shown below. The circuit has two inputs, A and B, and one output, Z. The inputs are assumed to be periodic as shown below. Assume that $R_{ON}$ for each MOSFET is zero.
(a) Sketch and clearly label the waveform for the output \( Z \) for \( 0 \leq t \leq T_4 \). In doing so, assume that \( C_G \) and \( C_L \) are both zero.

**Answer:** The waveform for the output \( Z \) for \( 0 \leq t \leq T_4 \) is given below.

![Waveform Diagram](image)

(b) Derive the time-average static power consumed by the circuit in terms of \( V_S \), \( R_L \), \( T_1 \), \( T_2 \), \( T_3 \) and \( T_4 \). Here, time-average power is defined as the total energy dissipated by the gate during the period \( 0 \leq t \leq T_4 \) divided by \( T_4 \).

**Answer:** For \( 0 \leq t \leq T_1 \) only the first MOSFET is on, i.e.

\[
P_{\text{static}} = \frac{V_S^2}{R_L + R_{\text{on}}}
\]  

(13)

For \( T_1 \leq t \leq T_2 \) the first and the third MOSFET's are on, i.e.

\[
P_{\text{static}} = 2 \frac{V_S^2}{R_L + R_{\text{on}}}
\]  

(14)

For \( T_2 \leq t \leq T_3 \) again only the first MOSFET is on, i.e.

\[
P_{\text{static}} = \frac{V_S^2}{R_L + R_{\text{on}}}
\]  

(15)

For \( T_3 \leq t \leq T_4 \) only the second MOSFET is on, i.e.

\[
P_{\text{static}} = \frac{V_S^2}{R_L + R_{\text{on}}}
\]  

(16)
Therefore, the time-average static power consumed by the circuit is given by

\[
P_{\text{static,ave}} = \frac{T_1}{T_4} \left( \frac{V_s^2}{R_L + R_{on}} \right) + \frac{T_2 - T_1}{T_4} \left( \frac{2V_s^2}{R_L + R_{on}} \right) + \frac{T_3 - T_2}{T_4} \left( \frac{V_s^2}{R_L + R_{on}} \right)
\]

\[
+ \frac{T_1 - T_3}{T_4} \left( \frac{V_s^2}{R_L + R_{on}} \right)
\]

\[
= \frac{V_s^2}{R_L + R_{on}} \left( \frac{T_1 + T_2 + T_3}{T_4} \right)
\]

(17)

(c) Now assume that \( C_G \) and \( C_L \) are nonzero. Derive the time-average dynamic power consumed by the circuit in terms of \( V_s, R_L, C_G, C_L, T_1, T_2, T_3 \) and \( T_4 \). In doing so, assume that the circuit time constants are all much smaller than \( T_1, T_2 - T_1, T_3 - T_2 \) and \( T_4 - T_3 \).

**Answer:** For \( 0 \leq t \leq T_1 \) the dynamic dissipation occurs while \( C_G \) discharges, and \( C_L \) charges, i.e.

\[
P_{\text{dynamic}} = \frac{C_G V_s^2 + C_G V_s^2}{2T_1}
\]

(18)

For \( T_1 \leq t \leq T_2 \) the dynamic dissipation occurs while \( C_L \) discharges, i.e.

\[
P_{\text{dynamic}} = \frac{C_L V_s^2}{2(T_2 - T_1)}
\]

(19)

For \( T_2 \leq t \leq T_3 \) the dynamic dissipation occurs while \( C_L \) charges, i.e.

\[
P_{\text{dynamic}} = \frac{C_L V_s^2}{2(T_3 - T_2)}
\]

(20)

For \( T_3 \leq t \leq T_4 \) the dynamic dissipation occurs while \( C_L, C_G \) charges, and \( C_L \) discharges, i.e.

\[
P_{\text{dynamic}} = \frac{C_G V_s^2 + C_G V_s^2}{2(T_4 - T_3)}
\]

(21)

Thus, the time-average dynamic power consumed by the circuit is given by

\[
P_{\text{dynamic,ave}} = \frac{T_1}{T_4} \left( \frac{C_G V_s^2 + C_G V_s^2}{2T_1} \right) + \frac{T_2 - T_1}{T_4} \left( \frac{C_L V_s^2}{2(T_2 - T_1)} \right) + \frac{T_3 - T_2}{T_4} \left( \frac{C_L V_s^2}{2(T_3 - T_2)} \right)
\]

\[
+ \frac{T_1 - T_3}{T_4} \left( \frac{C_G V_s^2 + C_G V_s^2}{2(T_4 - T_3)} \right)
\]

\[
= \frac{V_s^2}{T_4} (C_G + 2C_L)
\]

(22)

(d) Evaluate the time-average static and dynamic powers for \( V_s = 5 \) V, \( R_L = 10 \) k\( \Omega \), \( C_G = 100 \) fF, \( C_L = 1 \) pF, \( T_1 = 100 \) ns, \( T_2 = 200 \) ns, \( T_3 = 300 \) ns and \( T_4 = 600 \) ns.
Answer:

\[ P_{\text{static,ave}} = \frac{V_S^2}{R_L + R_{on}} \left( -\frac{T_1 + T_2 + T_4}{T_4} \right) \]
\[ = \frac{10 \times 10^3 + 0}{600} \left( -\frac{100 + 200 + 600}{600} \right) \]
\[ = 2.9 mW \]  

\[ P_{\text{dynamic,ave}} = \frac{1}{T_4} \left( C_G V_S^2 + 2 C_L V_S^2 \right) \]
\[ = \frac{1}{600 \times 10^{-9}} \left( 100 \times 10^{-15} \cdot 5^2 + 2 \cdot 1 \times 10^{-12} \cdot 5^2 \right) \]
\[ = 87.5 \mu W \]  

(e) By what percentage does the total time-average power consumption drop if the power supply voltage \( V_S \) drops by 30%?

**Answer:** Since the power depends linearly on \( V_S^2 \), a 30% drop in \( V_S \) translates to a 51% drop in the total time-average power consumption.

**Problem 7-4:** This problem studies the modeling of lumped-parameter thermal systems with electrical networks comprised of sources, resistors and capacitors. Here, voltage represents temperature, and current represents the flow of heat, or thermal energy.

(a) As a specific example, consider the thermal behavior of a house modeled as a single thermal unit. Heat is supplied to the house by its furnace, which is modeled by a current source. The heat is stored in the heat capacity of the house, which is modeled by a capacitor. Heat conducts from the house to the air outside through the windows, outer walls and roof of the house; heat conduction through the foundation is ignored here. The path of heat conduction through the walls and roof is modeled by a resistor, and the path of heat conduction through the windows is modeled by another resistor. The outside air temperature is set by a voltage source from the reference of absolute zero temperature. Following this description, derive a three-node electrical network which models the thermal behavior of the house. Two node voltages should represent the temperatures inside and outside the house, respectively. The third node should be the ground node at absolute zero temperature. Clearly identify each node and the thermal correspondence of each electrical element. In the electrical model, let a 1-V difference between nodes correspond to a 1°C temperature rise, and let a 1-A branch current correspond to a 1-W heat flow.

**Answer:** The circuit representation of the thermal behavior of a house is shown below.

(b) Let the house be 8 m wide, 20 m long and 8 m high, and approximate its roof as being flat. Further, let the house have 36 windows each with an area of 1 m². Given that the thermal conductivity of the walls and roof is 0.2 W/m²°C, and the thermal conductivity of the windows is 2.0 W/m²°C, determine the electrical resistance of the two resistors in the model.
**Answer:** The combined resistance of wall and roof is computed by

\[ R_{\text{wall \\& roof}} = \{0.2[2(8 \cdot 20) + 2(8 \cdot 8) + 8 \cdot 20 - 36]\}^{-1} \]

\[ = 8.74 \times 10^{-3} \left[ \frac{\text{C}}{\text{W}} \right] \]  

(24)

Similarly,

\[ R_{\text{window}} = (2 \cdot 36)^{-1} \]

\[ = 13.9 \times 10^{-3} \left[ \frac{\text{C}}{\text{W}} \right] \]  

(25)

(c) If the outside air temperature is 0 °C, and the furnace is turned off, then the temperature inside the house will fall from 20 °C to 10 °C in 12 hours. Determine the electrical capacitance of the capacitor in the model.

**Answer:** Let \( R_{\text{EQ}} = R_{\text{wall \\& roof}} \parallel R_{\text{window}}. \) Because the outside air temperature is 0 °C, and the furnace is turned off, both the current and voltage sources are turned off in the circuit representation of the thermal behavior of house. Thus,

\[ v_C(t) = V_0 e^{-\frac{t}{\tau}} \]  

(26)

Since, \( v_C(t = 0) = 20 \) and \( v_C(t = 10) = 10, \) Eq. (26) reduces to

\[ \tau = R_{\text{EQ}} C \]

\[ = \frac{10}{\ln 2} \]

\[ = 14.427 \]  

(27)

\[ \Rightarrow C = \frac{1}{R_{\text{EQ}}} \left( \frac{10}{\ln 2} \right) = 2.7931 \times 10^3 \left[ \frac{\text{J}}{\text{C}°} \right] \]

(d) Given an outside air temperature of -10 °C, and a temperature inside the house of 20 °C, determine the rate at which the furnace must supply heat to the house in order to maintain the temperature difference in steady state.

**Answer:** In steady state the capacitor looks like an open circuit. Since the temperature across the resistors is to be maintained at 30°C,

\[ I = \frac{30}{R_{\text{EQ}}} \]

\[ = 5.808 \times 10^3 \text{[W]} \]  

(28)
(e) Overnight, the temperature inside the house is allowed to cool to 10 °C. In the morning, the outside air temperature is 0 °C. Determine the constant rate at which the furnace must supply heat in order to raise the temperature inside the house to 20 °C in 5 minutes.

**Answer:** Combining the ZSR and ZIR of capacitor voltage yield

\[ v_C(t) = I R_{EQ} \left( 1 - e^{-\frac{t}{R_C}} \right) + V_0 e^{-\frac{t}{R_C}} \]  \hspace{1cm} (29)

Since \( V_0 = 10 \) and \( v_C(t = \frac{1}{15}) = 20 \),

\[ 20 = I \cdot 5.2 \times 10^{-3} \left( 1 - e^{-\frac{1}{15 \cdot 14.427}} \right) + 10 e^{-\frac{1}{15 \cdot 14.427}} \]

\[ \Rightarrow I = \frac{10}{5.2} \left( \frac{2 - e^{-\frac{1}{15 \cdot 14.427}}}{1 - e^{-\frac{1}{15 \cdot 14.427}}} \right) = 335.8[W] \] \hspace{1cm} (30)

(f) Assume that the house is in the thermal steady state described in Part (d). At \( t = 0 \), the outside air temperature steps to 0°C, while the furnace continues to supply heat at the same rate. Determine the temperature in the house for \( t \geq 0 \).

**Answer:** Separating the impact of sudden increase in outside temperature as a step input through an additional voltage source allows for applying superposition principle to the circuits shown below.

![Circuit Diagram](image)

Applying superposition, the first circuit yields,

![Circuit Diagram](image)

In this circuit, \( v_C \) is fixed at 20°C because the current source is supplying the heat that would maintain the inside of the house at that temperature as solved in (d).
For the second voltage source, we have the following topology.

\[ v_c(t) \quad \frac{R_{\text{wall \& roof}}}{\quad C} \quad \frac{R_{\text{window}}}{\quad v(t)} \quad 10 \quad t \]

The voltage across the capacitor, \( v_C \) which correspond to the temperature of the house, is given by

\[ v_C = 10(1 - e^{-\frac{t}{\tau}}) \quad (31) \]

Thus, the overall temperature of the house \( v_C(t) \) for \( t \geq 0 \) is given by

\[ v_C = 10(3 - e^{-\frac{t}{\tau}}) \quad (32) \]