Exercise 8-1: Find the ZIR, the ZSR and the total response of the states in the two networks shown below for $t > 0$.

Answer:
For $t > 0$

- The response of capacitor circuit

\[
\begin{align*}
    v_{\text{ZIR}}(t) &= V_0 e^{-\frac{t}{\tau}} \\
    v_{\text{ZSR}}(t) &= I_0 R_1 \left(1 - e^{-\frac{t}{\tau}}\right) \\
    v_{\text{total}}(t) &= v_{\text{ZIR}}(t) + v_{\text{ZSR}}(t) \\
        &= V_0 e^{-\frac{t}{\tau}} + I_0 R_1 \left(1 - e^{-\frac{t}{\tau}}\right)
\end{align*}
\]

where $\tau = (R_1 + R_2)C$
The response of inductor circuit

\[ i_{ZIR}(t) = I_0 e^{-\frac{t}{\tau}} \]
\[ i_{ZSR}(t) = V_0 R_1 \left( 1 - e^{-\frac{t}{\tau}} \right) \]

\[ i_{\text{total}}(t) = i_{ZIR}(t) + i_{ZSR}(t) = I_0 e^{-\frac{t}{\tau}} + \frac{V_0}{R_1} \left( 1 - e^{-\frac{t}{\tau}} \right) \]

where \( \tau = \frac{L}{R_1 || R_2} \)

**Problem 8-1:** In the network shown below, the inductor and capacitor have zero states at \( t = 0^- \). At \( t = 0 \), a step in current from 0 to \( I_0 \) is applied by the current source as shown.

(a) Find \( v \) and \( dv/dt \) at \( t = 0^+ \).

**Answer:**

Using continuity conditions

\[ v(0^+) = v(0^-) = 0 \]

\[ \frac{dv}{dt}(0^+) = \frac{1}{C}i(0^+) = \frac{1}{C}i(0^-) = 0 \]

(b) Argue that \( i = 0 \) at \( t = \infty \), and then find \( v \) at \( t = \infty \).

**Answer:**

At \( t = \infty \) a capacitor looks like an open circuit and an inductor looks like a short circuit. Thus, when a capacitor and an inductor are connected in series, the current through inductor becomes zero at \( t = \infty \).

For the given circuit

\[ v(t = \infty) = I_0 R \]

(c) Find a second-order differential equation which describes the behavior of \( v(t) \) for \( t \geq 0^+ \).

**Answer:**

Applying KCL

\[ I(t) = \frac{1}{R} \left( v + L \frac{di}{dt} \right) + i \]

\[ = \frac{1}{R} \left( v + L C \frac{d^2v}{dt^2} \right) + \frac{C}{R} \frac{dv}{dt} \quad \text{since} \quad i = C \frac{dv}{dt} \]

\[ \implies \quad \frac{d^2v}{dt^2} + \frac{R}{L} \frac{dv}{dt} + \frac{1}{LC} v = \frac{R}{LC} I_0 \]
(d) The voltage \( v(t) \) takes the form \( v(t) = V_1 + V_2 \cos(\omega t + \phi)e^{-\alpha t} \) for \( t \geq 0^+ \). Find \( V_1, V_2, \omega, \phi \) and \( \alpha \). Also find \( i(t) \).

**Answer:**
Let \( \omega_\circ = \frac{1}{\sqrt{LC}} \) and \( \alpha = \frac{R}{2L} \). Then

\[
\omega = \sqrt{\omega_\circ^2 - \alpha^2}
\]  

(6)

Since \( v(t = \infty) = I_0R \),

\[
V_1 = I_0R
\]  

(7)

Given \( v(0) = 0; \frac{dv}{dt}(0) = 0 \), i.e.

\[
0 = I_0R + V_2 \cos \phi
\]
\[
0 = -\alpha V_2 \cos \phi - \omega V_2 \sin \phi,
\]

\[\Rightarrow \quad \phi = -\arctan \left( \frac{\alpha}{\omega} \right)\]

(8)

\[
V_2 = -\frac{I_0R}{\cos \phi} = -I_0R\frac{\sqrt{\omega^2 + \alpha^2}}{\omega} = -I_0R\omega_\circ \omega
\]

since \( \cos \phi = \frac{\omega}{\sqrt{\omega^2 + \alpha^2}} \)

Finally,

\[
i(t) = C \frac{dv}{dt}
\]
\[
= C \left[ -\frac{\alpha V_2 \cos(\omega t + \phi)e^{-\alpha t} - \omega V_2 \sin(\omega t + \phi)e^{-\alpha t}}{\cos \phi} \right]
\]
\[
= I_0RC \frac{\omega_\circ}{\omega} e^{-\alpha t} \left[ \alpha \cos(\omega t + \phi) + \omega \sin(\omega t + \phi) \right]
\]

(9)

(e) Sketch and clearly label \( v(t) \) and \( i(t) \) for \( t \geq 0 \) over several sinusoidal cycles assuming \( \omega \approx 10\alpha \).

**Answer:** For \( \omega \gg \alpha \)

\[
v(t) = I_0R \left[ 1 - \frac{\omega_\circ}{\omega} \cos(\omega t + \phi)e^{-\alpha t} \right]
\]
\[
\approx I_0R \left[ 1 - \cos(\omega t + \phi)e^{-\alpha t} \right]
\]

(10)

\[
i(t) = I_0RC \frac{\omega_\circ}{\omega} e^{-\alpha t} \left[ \alpha \cos(\omega t + \phi) + \omega \sin(\omega t + \phi) \right]
\]
\[
\approx I_0R \sqrt{\frac{C}{L}} e^{-\alpha t} \sin(\omega t + \phi)
\]
where $T = \frac{2\pi}{\omega}$.

(f) Under the condition of Part (e), sketch and clearly label $i(t)$ versus $v(t)$ with $t$ acting as a parameter along the sketch.

Answer:

(g) Suppose that the input $I(t)$ is a current impulse with area $Q_o$. Find the response of the network to this input. Before solving this problem by brute force, consider your answer to Problem 6.4.

Answer:
Since \( Q_0 \cdot \delta(t) = \frac{Q_0}{I_0} \frac{d}{dt} (I_0 \cdot u(t)) \),

\[
\begin{align*}
\nu_{\text{impulse}} &= \frac{Q}{I_0} \frac{d}{dt} \nu_{\text{step}}(t) \\
\nu_{\text{impulse}} &= \frac{Q}{I_0} \frac{d}{dt} i_{\text{step}}(t) \\
\end{align*}
\]

(11)

Problem 8-2: The network shown below includes two switches: #1 and #2. Prior to \( t = 0 \), both switches are closed, and the capacitor voltage \( \nu(t) \) and inductor current \( i(t) \) are both zero.

(a) At \( t = 0 \), Switch #1 opens, and it remains open until \( t = T_1 \). Find \( i(t) \) and \( \nu(t) \) for \( 0 \leq t \leq T_1 \).

Answer: For \( 0 \leq t \leq T_1 \)
\[ v(t) = \frac{1}{C} \int_0^t I \, d\tau = \frac{I}{C} t \]  
\tag{12} \]

\[ i(t) = 0 \quad \text{since the inductor remains shorted} \]

(b) At \( t = T_1 \), Switch \#1 closes as Switch \#2 simultaneously opens. They remain in these states until \( v(t) \) goes to zero, at which time Switch \#2 closes. Define the time at which \( v(t) \) goes to zero as \( t = T_2 \). Determine \( T_2 \), and find \( v(t) \) and \( i(t) \) for \( T_1 \leq t \leq T_2 \).

**Answer:** Let \( \omega = \frac{1}{\sqrt{LC}} \).

For \( T_1 \leq t \leq T_2 \)

\[ v(t) = \frac{IT_1}{C} \cos [\omega (t - T_1)] \]

\[ i(t) = -C \frac{dv}{dt} = \omega IT_1 \sin [\omega (t - T_1)] \]  
\tag{13} \]

Since \( v(t) = 0 \) at \( T_2 \)

\[ 0 = v(T_2) \]

\[ = \frac{IT_1}{C} \cos [\omega (T_2 - T_1)] \]

\[ \Rightarrow \quad T_2 = \frac{\pi}{2\omega} + T_1 \]  
\tag{14} \]

(c) Both switches remain closed until \( t = T_3 \). Find \( v(t) \) and \( i(t) \) for \( T_2 \leq t \leq T_3 \).

**Answer:** For \( T_2 \leq t \leq T_3 \)

\[ v(t) = 0 \]

\[ i(t) = \omega IT_1 \]  
\tag{15} \]

as both the capacitor and the inductor gets shorted.
(d) At $t = T_3$, Switch #1 again opens, and it remains open until $t = T_4$. Find $v(t)$ and $i(t)$ for $T_3 \leq t \leq T_4$.

**Answer:** Similar to part (a) for $T_3 \leq t \leq T_4$

\[
v(t) = \frac{I}{C}(t - T_3)
\]

\[
i(t) = \omega IT_1
\]

(e) Finally, at $t = T_4$, Switch #1 closes as Switch #2 again simultaneously opens. They remain in these states until $v(t)$ again goes to zero, at which time Switch #2 closes. Define the time at which $v(t)$ again goes to zero as $T_5$. Determine $T_5$, and find $v(t)$ and $i(t)$ for $T_4 \leq t \leq T_5$.

**Answer:** For $T_4 \leq t \leq T_5$

\[
v(t) = I \sqrt\frac{(T_4 - T_3)^2 + T_1^2 \cos(\omega(t - T_4) + \phi)}{C}
\]

\[
i(t) = I \omega \sqrt(T_4 - T_3)^2 + T_1^2 \sin(\omega(t - T_4) + \phi)
\]

where

\[
\phi = \arctan\left(\frac{T_1}{T_4 - T_3}\right)
\]

Since $v(t) = 0$ at $T_5$

\[
0 = v(T_5)
\]

\[
= I \sqrt\frac{(T_4 - T_3)^2 + T_1^2 \cos(\omega(T_5 - T_4) + \phi)}{C}
\]

\[
\implies \omega(T_5 - T_4) + \phi = \frac{\pi}{2}, \text{ i.e.}
\]

\[
T_5 = \frac{1}{\omega} \left[\frac{\pi}{2} - \arctan\left(\frac{T_1}{T_4 - T_3}\right)\right] + T_4
\]
(f) Sketch and clearly label \( v(t) \) and \( i(t) \) for \( 0 \leq t \leq T_5 \).

Answer:

\[ v(t) \quad T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad i(t) \]

(g) Sketch and clearly label a graph of \( i(t) \) versus \( v(t) \) with time as a parameter along the graph for \( 0 \leq t \leq T_5 \). Indicate on the graph the points at which \( t = 0, T_1, T_2, T_3, T_4 \) and \( T_5 \). Also identify those portions of the graph which correspond to contours of constant stored energy within the system.

Answer:
Problem 8-3: This problem is a continuation of Problem 8-2. It explores the use of energy conservation to analyze the operation of the network described therein.

(a) Determine the energy stored in the capacitor at \( t = T_1 \).

Answer:
Since \( v(T_1) = \frac{IT_1}{C} \)

\[
E_{\text{cap}}(T_1) = \frac{1}{2} C v^2(T_1) = \frac{1}{2C} I^2 T_1^2
\]  
(20)

(b) The energy stored in the capacitor at \( t = T_1 \) is transferred to the inductor at \( t = T_2 \). Use this fact to determine \( i(T_2) \). Your answer here should match your answer to Part (b) of Problem 8-2.

Answer:
Since \( E_{\text{ind}}(T_2) = E_{\text{cap}}(T_1) = \frac{1}{2C} I^2 T_1^2 \)

\[
E_{\text{ind}}(T_2) = \frac{1}{2} L i^2(T_2) = \frac{1}{2C} I^2 T_1^2
\]

\[\implies i(T_2) = \frac{1}{\sqrt{LC}} IT_1 = \omega IT_1 \]  
(21)

Note: depending on how the direction of current is defined, the current can be either positive or negative when taking the square roots.

(c) Determine the energy stored in the capacitor at \( t = T_4 \).

Answer:
Similar to part (a) since \( v(T_4) = \frac{I(T_4 - T_3)}{C} \)

\[
E_{\text{cap}}(T_4) = \frac{1}{2C} I^2 (T_4 - T_3)^2
\]  
(22)

(d) Use energy conservation to determine the energy stored in the inductor at \( t = T_5 \), and then determine \( i(T_5) \). Your answer here should match your answer to Part (c) of Problem 8-2.

At \( t = T_5 \), all the energy stored in the capacitor at \( t = T_4 \) is transferred and added to the energy stored in the inductor, i.e.

\[
E_{\text{ind}}(T_5) = \frac{1}{2C} I^2 (T_4 - T_3)^2 + \frac{1}{2C} I^2 T_1^2
\]

\[= \frac{I^2}{2C} [(T_4 - T_3)^2 + T_1^2] \]  
(23)

Since \( E_{\text{ind}}(T_5) = \frac{1}{2} L i^2(T_5) \)

\[
E_{\text{ind}}(T_5) = \frac{1}{2} L i^2(T_5) = \frac{I^2}{2C} [(T_4 - T_3)^2 + T_1^2]
\]

\[\implies i(T_5) = \frac{I}{\sqrt{LC}} \sqrt{(T_4 - T_3)^2 + T_1^2} = \omega I \sqrt{(T_4 - T_3)^2 + T_1^2} \]  
(24)
(e) Now let the switches move repetitively through the cycle in which they both begin closed, then Switch #1 opens, then both switches simultaneously change state, then Switch #2 closes to return to the beginning of the cycle. Assume that in each cycle Switch #1 is open alone for the duration \( T \). Further, assume that Switch #2 always closes when \( v(t) \) reaches zero. Assuming that \( v \) and \( i \) are initially zero, Determine \( i \) at the end of the \( n \)th switching cycle in terms of \( n, C, L, T \) and \( I \).

**Answer:**
At the end of each cycle, the energy, \( E = \frac{I^2T^2}{2C} \) is delivered from the current source to the inductor via capacitor. Thus,

\[
\Delta i^2 = \frac{I^2T^2}{LC} \quad (25)
\]

at the end of each cycle, or

\[
i[n] = \frac{\sqrt{nIT}}{\sqrt{LC}} \quad (26)
\]

at the end of \( n \)th switching cycle.