Exercise 9-1: The network shown below begins operation at rest at \( t = 0^- \). At \( t = 0 \), both sources apply an impulse to the network. The voltage source impulse has area \( \Lambda \), and the current source impulse has area \( Q \). Determine \( v(t) \) and \( i(t) \) for \( t > 0^- \).

Answer: This network consists of two energy storage elements and two impulse sources, so immediately we know that the response will 'ring' between the elements with initial conditions imposed by the impulses and, furthermore, that the response will not decay since there is no resistor (i.e. no dissipation). Since there are two sources, superposition is an appropriate solution method:

\[
\begin{align*}
  i(t) &= i_v(t) + i_i(t) \\
  v(t) &= v_v(t) + v_i(t)
\end{align*}
\]  

where the subscripts indicate the source being considered for that component of the total solution.

Voltage Source Solution:

For the voltage source solution the current source is replaced by an open circuit. For the rapid changes in voltage produced by the voltage impulse, the capacitor is approximated by a short circuit (S.C.) so the voltage impulse will deliver a step change in current through the inductor. The initial conditions must then be

\[
\begin{align*}
  i_v(0^+) &= \frac{\Lambda}{L} \\
  v_v(0^+) &= 0
\end{align*}
\]  

The general form of the second order solution due to an impulse with no exponential decay term is
\[ v_v(t) = V_o \sin(wt + \phi) \]
\[ i_v(t) = C \frac{dv(t)}{dt} = CV_o \omega \cos(wt + \phi) \] (3)

where, by inspection, the natural frequency is
\[ \omega = \frac{1}{\sqrt{LC}} \] (4)

Evaluating the current and voltage at the initial condition gives
\[ v_v(0^+) = 0 = V_o \sin(\phi) \]
\[ i_v(0^+) = \frac{\Lambda}{L} = CV_o \omega \cos(\phi) \] (5)

The two unknowns can then be solved for:
\[ \phi = 0 \]
\[ V_o = \frac{\Lambda}{LC \omega} = \frac{\Lambda}{\sqrt{LC}} \] (6)

The final solution for the response due to the voltage source is then
\[ v_v(t) = \frac{\Lambda}{\sqrt{LC}} \sin(t/\sqrt{LC}) \]
\[ i_v(t) = \frac{\Lambda}{L} \cos(t/\sqrt{LC}) \] (7)

Current Source Solution:

The solution due to the current source alone is very similar. The voltage source is replaced by an open circuit. For 'the rapid changes in current produced by the current impulse the inductor by an open circuit (O.C.) so the voltage impulse will deliver a step change in voltage across the inductor and capacitor which are now in parallel. The initial conditions must then be
\[ i_i(0^+) = 0 \]
\[ v_i(0^+) = \frac{Q}{C} \] (8)

Again a valid general form of the solution is
\[ i_i(t) = I_o \sin(wt + \phi) \]
\[ v_i(t) = -L \frac{di_i(t)}{dt} = -LI_o \omega \cos(wt + \phi) \] (9)

with the same natural frequency.
Evaluating the current and voltage at the initial condition gives

\[ i_I(0^+) = 0 = I_o \sin(\phi) \]
\[ v_I(0^+) = \frac{Q}{C} = -LI_o \omega \cos(\phi) \]  

(10)

The two unknowns can then be solved for:

\[ \phi = 0 \]
\[ I_o = -\frac{Q}{LC \omega} = -\frac{Q}{\sqrt{LC}} \]  

(11)

The final solution for the response due to the current source is then

\[ i_I(t) = -\frac{Q}{\sqrt{LC}} \sin(t/\sqrt{LC}) \]
\[ v_I(t) = \frac{Q}{C} \cos(t/\sqrt{LC}) \]  

(12)

The superposition of the responses due to each source gives the response for the complete circuit

\[ i(t) = \frac{\Lambda}{L} \cos(t/\sqrt{LC}) - \frac{Q}{\sqrt{LC}} \sin(t/\sqrt{LC}) \]
\[ v_I(t) = \frac{\Lambda}{\sqrt{LC}} \sin(t/\sqrt{LC}) + \frac{Q}{C} \cos(t/\sqrt{LC}) \]  

(13)

**Problem 9-1:** The two networks shown below are driven in sinusoidal steady state by the voltage \( v_I(t) = V_I \cos(\omega t) \). Their outputs take the form \( v_O(t) = V_O \cos(\omega t + \phi) \).

![Network Diagrams](image)

(A) For both networks, find \( V_O \) and \( \phi \) as functions of \( V_I \) and \( \omega \) using impedance methods.

**Answer:** Using impedance methods, both of these circuits represent simple voltage dividers. Furthermore we replace the original source by a complex input \( \tilde{V}_I e^{j\omega t} \) and solve for the corresponding complex output amplitude \( \tilde{V}_O \) where these quantities are related to the original problem by
\[ v_i(t) = Re\{\tilde{v}_i e^{j\omega t}\} = V_i \cos(\omega t) \]
\[ v_o(t) = Re\{\tilde{v}_o e^{j\omega t}\} = V_o \cos(\omega t + \phi) \]  

Network #1:

The voltage divider for Network #1 is simplified as follows

\[ \tilde{v}_o = \frac{sL + 1/sC}{R + sL + 1/sC} \tilde{v}_i \]
\[ = \frac{sLC + 1}{s^2LC + sRC + 1} V_i \]
\[ = \frac{1 - \omega^2 LC}{1 - \omega^2 LC + j\omega RC} V_i \]
\[ = \frac{|1 - \omega^2 LC|}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}} V_i e^{j[\frac{\pi}{2} - \tan^{-1}(\omega RC/(1 - \omega^2 LC))]}. \]

which is now in the form of a real amplitude times a complex phase

\[ V_o = V_i \frac{|1 - \omega^2 LC|}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}} \]
\[ \phi = -\tan^{-1}(\omega RC/(1 - \omega^2 LC)) \]

The original output is reconstructed by multiplying by \( e^{j\omega t} \) and taking the real part.

Network #2:

Use the same procedure where the voltage divider output is now across the parallel L and C components.

\[ \tilde{v}_o = \frac{(sL/sC)}{(sL/sC) + R(sL + 1/sC)} \tilde{v}_i \]
\[ = \frac{sL/R}{s^2LC + sL/R + 1} V_i \]
\[ = \frac{j\omega L/R}{1 - \omega^2 LC + j\omega L/R} V_i \]
\[ = \frac{\omega L/R}{\sqrt{(1 - \omega^2 LC)^2 + (\omega L/R)^2}} V_i e^{j[\frac{\pi}{2} - \tan^{-1}(\omega L/R)/(1 - \omega^2 LC)]}. \]

\[ V_o = V_i \frac{\omega L/R}{\sqrt{(1 - \omega^2 LC)^2 + (\omega L/R)^2}} \]
\[ \phi = \pi/2 - \tan^{-1}(\omega L/R)/(1 - \omega^2 LC)) = \tan^{-1}(1 - \omega^2 LC)/(\omega L/R) \]
(B) For both networks, let $R = 1000\Omega$, $L = 47mH$ and $C = 4.7nF$. Plot and clearly label $V_o / V_i$ for $2\pi \times 10^5 \text{rad/s} \leq \omega \leq 2\pi \times 10^5 \text{rad/s}$; use a linear axis for $V_o / V_i$, and a logarithmic axis for $\omega$. You need only plot enough points to outline the dependence $V_o / V_i$ on $\omega$.

**Answer:** You can easily create the magnitude sketch from a few features: the natural frequency, the amplitude at the natural frequency, the low frequency asymptote, the high frequency asymptote, and possibly the junction of the asymptotes. For Network #1

$$\omega_N = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(47e-3)(4.7e-9)}} = 1.07e4 \text{ rad/} \text{sec}$$

$$v_o(t) \approx V_i \text{ for } \omega \ll \omega_N \text{ or } \omega \gg \omega_N$$

$$v_o(1/\sqrt{LC}) = 0$$

and for Network #2

$$V_o = V_i \omega L/R \text{ for } \omega \ll \omega_N$$

$$V_o = V_i / RC \omega \text{ for } \omega \gg \omega_N$$

$$|\text{Asymptote Intersection}| = \left(\frac{V_i}{R}\right)\left(\frac{\sqrt{L}}{C}\right)$$

$$v_o(1/\sqrt{LC}) = V_i$$

![Problem 9-1(b) Network #1](image-url)
Problem 9-1(b) Network #2

(C) Describe the filtering function of each network, and how each network acts to perform its function.

**Answer:** Network #1 functions as a NOTCH filter attenuating the output voltage amplitude near the natural, or resonant, frequency of the circuit. Examining the specific circuit elements we know that the series L and C components can be approximated by open circuit and short circuit element in series for very low and very high frequencies. Thus since no current flows through the circuit in these frequency regions $v_o \approx v_i$.

Network #2 functions as a BAND-PASS filter attenuating the output voltage amplitude for very high and very low frequencies compared to the resonant frequency of the circuit. Using the same open and short circuit approximations, connecting the L and C in parallel means that outside the band near the resonant frequency one of these elements will act as a short circuit so that $v_o = 0$.

**Problem 9-2:** This problem examines the very simple tuner for an AM radio shown below. Here, the tuner is the parallel inductor and capacitor. The injection of radio signals into the tuner by the antenna is modeled by a current source, while the Norton resistance of the antenna in parallel with the remainder of the radio is modeled by a resistor. (You will learn about antenna modeling in 6.014.) The AM radio band extends from 540 kHz through 1600 kHz. The information transmitted by each radio station is constrained to be within ±5 kHz of its center frequency. (You will learn about AM radio transmission in 6.003.) To prevent frequency overlap of neighboring stations, the center frequency of each station is constrained to be a multiple of 10 kHz. Therefore, the purpose of the tuner is to pass all frequencies within 5 kHz of the center frequency of the selected station, while attenuating all other frequencies.
(A) Assume that \( I(t) = I \cos(\omega t) \). Find \( v(t) \) where \( v(t) = V \cos(\omega t + \phi) \), and both \( v(t) \) and \( \phi \) are functions of \( \omega \). Note that \( v(t) \) is the output of the tuner, namely the signal that is passed on to the remainder of the radio.

**Answer:** This is also solved using impedance methods as in Problem 9-1(B). This answer looks remarkably like the P9-1(B), Network #2, BAND-PASS circuit solution (hint: consider the Norton equivalent of P9-1(B), Network #2).

\[
\tilde{v} = \frac{1}{(1/sL) + sC + 1/R} \tilde{I} \\
= \frac{sL}{s^2LC + sL/R + 1} I \\
= \frac{j\omega L}{1 - \omega^2LC + j\omega L/R} I \\
= \frac{\omega L}{\sqrt{(1 - \omega^2LC)^2 + (\omega L/R)^2}} I e^{j[\pi/2 - \tan^{-1}(\omega L/R)/(1 - \omega^2LC)]}
\]

\[
V = \frac{\omega L}{\sqrt{(1 - \omega^2LC)^2 + (\omega L/R)^2}}
\]

\[
\phi = \pi/2 - \tan^{-1}(\omega L/R)/(1 - \omega^2LC)) = \tan^{-1}(1 - \omega^2LC)/(\omega L/R))
\]

(B) For a given combination of \( I, C, L \) and \( R \), at what frequency is \( V \) maximized?

**Answer:** \( V \) is maximized when the highest order term of \( \omega \) in the denominator is minimized which, like the previous problem, is at

\[
\omega_{\text{max}} = \frac{1}{\sqrt{LC}}
\]

(C) Assume that \( L = 365 \, \mu H \). Over what range of capacitance must \( C \) vary so that the frequency of maximum \( V/I \) may be tuned over the entire AM band? Not that tuning the frequency of maximum \( V/I \) to the center frequency of a particular station tunes in that station.

**Answer:**
As a compromise between passing all frequencies within 5 kHz of a center frequency and rejecting all frequencies outside that band, let the design criterion, determine \( R \).

**Answer:** First the amplitude at the center frequency is found by substituting the center frequency found in part (B) into the amplitude function found in part (A). The result as expected is just

\[
V(\omega_N) = IR
\]

(25)

The frequency 5kHz away can be represented as a fraction, \( \gamma \), of the center frequency

\[
\omega = \gamma \omega_N = \gamma / \sqrt{LC}
\]

(26)

Substituting this into the amplitude function from part (A) and dividing the numerator and denominator by \( \omega \) gives

\[
V(\gamma \omega_N) = I \frac{L}{\sqrt{LC(1/\gamma - \gamma)^2 + (L / R)^2}}
\]

(27)

The amplitude ratio with respect to the center frequency is defined as

\[
\frac{V(\gamma \omega_N)}{V(\omega_N)} = \beta \leq \frac{L}{\sqrt{R^2 LC(1/\gamma - \gamma)^2 + L^2}}
\]

(28)

Rearranging and solving for \( R \) gives

\[
R \leq \frac{L(1 - \beta)}{\sqrt{\beta C(1/\gamma - \gamma)^2}}
\]

(29)

Substituting in the circuit element values and \( \beta = 0.25 \) and \( \gamma = 0.995 \) for the given amplitude ratio at 0.995 MHz (which is a tighter constraint on \( R \) than 1.005 MHz) gives

\[
R \leq 397 \text{ k}\Omega
\]

(30)

(E) Given your design for \( R \), determine \( V(1 \text{ MHz } \pm 10 \text{ kHz}) / V(1 \text{ MHz}) \). Also, determine \( Q \) for the tuner and its load resistor when the tuner is tuned to 1 MHz.
Answer: Use the same ratio equation as above but now with $\gamma = 0.99$ which results in

$$\frac{V(0.99\omega_N)}{V(\omega_N)} = 0.017$$ (31)

The quality factor associated with this design is defined as

$$Q = \frac{\omega_N}{2\alpha} = \frac{RC}{\sqrt{LC}} = R_1 \sqrt{\frac{C}{L}} = 173$$ (32)

**Problem 9-3** This problem explores the Thevenin and Norton equivalence of networks operating in sinusoidal steady state. The resistors, capacitors and inductors in these networks are all linear, and the voltage and current sources in these networks all operate at the same frequency.

(A) Determine the relations between $V_T$, $Z_T$, $\phi_T$, $I_N$, $Z_N$ and $\phi_N$ which must exist so that the $v - i$ relations for Networks #1 and #2 are identical at their terminals under sinusoidal steady state operation. In doing so, treat $V_T$, $\phi_T$, $I_N$ and $\phi_N$ as real constants. Note that it is also assumed that the terminal variables $v$ and $i$ operate in sinusoidal steady-state at the frequency $\omega$.

**Answer:** For the two circuits to be equivalent, if the same test voltage is applied across each terminal pair the same terminal current should be observed in each circuit. We can analyze the circuits using complex exponential sources and test voltages recognizing that the real part of each would represent the cosine equivalent sources shown and corresponding test voltages.

First the equivalent impedance seen at the terminals is observed by setting the Thevenin and Norton sources to zero. In this case, it is observed that

$$Z_T = Z_N = |Z|e^{j\phi}$$ (33)

which is complex and therefore has both magnitude and phase. Now assume a complex test voltage, $\tilde{V}e^{j\omega t}$, is placed across the terminals. Equating the current observed by each circuit results in
\[ i e^{j\omega} = \frac{\tilde{V} e^{j\alpha} - V_N e^{j(\alpha + \phi_N)}}{Z_T} = \frac{\tilde{V} e^{j\alpha}}{Z_N} - I_N e^{j(\alpha + \phi_N)} \]  

(34)

Since the equivalent impedances are equal, multiply through by \( Z_T e^{-j\omega} \) and then subtract \( \tilde{V} \) from each side to get

\[-V_T e^{j\beta_T} = -ZI_N e^{j\beta_N} = -|Z| e^{j\beta_N} I_N e^{j\beta_N} \]  

(35)

The magnitude and phase components can now be equated separately as

\[ V_T = |Z| I_N \]  
\[ \phi_T = \phi_Z + \phi_N \]  

(36)

and this analysis could be repeated with the same results if only real sources (by considering either the imaginary or real part) are considered.

(B) Review the corresponding arguments for networks involving only resistors and sources, and then briefly explain why Networks #1 and #2 may serve as the Thevenin and Norton equivalents, respectively, of an arbitrary network operating in sinusoidal steady state at the frequency \( \omega \).

Answer: Our original description of a Thevenin or Norton equivalent was that a network consisting of only ideal resistors and sources could be represented at a pair of terminals connected to the network by a single equivalent source and resistor. The corresponding definition for linear systems in the sinusoidal steady-state is that they can also be represented at a pair of terminals by a single source with real magnitude and a phase angle, and a complex impedance element which is a function of the steady-state frequency \( \omega \). Using impedance methods it has already been shown that more complex circuits involving these elements can be reduced using algebraic equations that are similar in nature to the tools used to develop the theory for just ideal resistors and sources.

(C) Determine \( V_T, Z_T, \) and \( \phi_T \) in the Thevenin equivalent of Network #3.

Answer: The Thevenin impedance is calculated by setting the source to zero which shows that the three impedances are in parallel just like in Problem 9-2(A)

\[ Z_T = \frac{sL}{s^2LC + sL/R + 1} \]  

(37)

The Thevenin complex voltage, \( \tilde{V} \), is the open circuit voltage across the parallel L and C elements.
\[
\tilde{v} = \frac{L/C}{R(sL + 1/sC) + L/C} V
\]
\[
= \frac{sL/R}{s^2LC + sL/R + 1} V
\]
\[
= \frac{j \omega L/R}{(1 - \omega^2LC) + j\omega L/R} V
\]
\[
= V_t e^{j\phi}
\]
\[
V_t = \frac{\omega L/R}{\sqrt{(1 - \omega^2LC)^2 + (\omega L/R)^2}} V
\]
\[
\phi = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega L/R}{1 - \omega^2LC}\right) = \tan^{-1}\left(\frac{1 - \omega^2LC}{\omega L/R}\right)
\]

(D) Suppose \(Z_T\) in Part C is to be implemented with either Network #4 or #5. Determine \(R_T\) and \(C_T\) in Network #4, and \(R_T\) and \(L_T\) in Network #5, in terms of \(R, L, C\), and \(\omega\). Hint: consider matching \(Z_T^{-1}\) rather than matching \(Z_T\). Under what circumstances is Network #4 preferred over Network #5, and vice versa?

Answer: The susceptance of Networks #4 and #5 are

\[
Z_4^{-1} = 1/R_T + j\omega C_T
\]
\[
Z_5^{-1} = 1/R_T + 1/j\omega L_T = 1/R_T - j/\omega L_T
\]

Their real and imaginary parts need to be matched to the Thevenin impedance calculated in (C) above

\[
Z_T^{-1} = sC + 1/R + 1/sL
\]
\[
= j\omega C + 1/R + 1/j\omega L
\]
\[
= 1/R + j(\omega C - 1/\omega L)
\]

For both networks

\[
R = R_T
\]

but the proper network choice is determined by the sign of the imaginary term. Since only positive-valued inductors and capacitors can be pulled off the shelf, if the imaginary term for \(Z_T\) is positive Network #4 should be implemented otherwise Network #5 should be chosen where

\[
C_T = \frac{\omega^2 LC - 1}{\omega^2 L} \quad \text{for } \omega^2 > 1/LC
\]
\[
L_T = \frac{L}{1 - \omega^2 LC} \quad \text{for } \omega^2 < 1/LC
\]