Exercise 10.1

Problem Set #10 Solutions SP 06.

What we seek is a frequency-independent voltage divider. We have:

\[ V_{\text{out}}(t) = V_{\text{out}} \cos(\omega t + \phi) \]

\[ V_{\text{in}}(t) \]

\[ V_{\text{in}} \]

\[ V_{\text{c1s}} \]

\[ V_{\text{out}} = \frac{R_2 (R_1 C_{1S} + 1)}{V_{\text{in}} (R_1 (R_2 C_{2S} + 1) + R_2 (R_1 C_{1S} + 1))} \]

We want \( \phi = 0^\circ \) for all \( \omega \) and \( V_{\text{out}} = \frac{R_2}{R_1 + R_2} V_{\text{in}} = -1 V_{\text{in}} \)

If we choose \( R_1 C_1 = R_2 C_2 \), then

\[ V_{\text{out}} = \frac{R_2}{V_{\text{in}} (R_1 + R_2)} V_{\text{in}} \]

\[ \therefore R_2 = R \]

\[ R_1 = 9R \]
Exercise 10.1 continued

\( \phi = 0^\circ \) means:

\[
\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R_2(R_1C_1s + 1)}{R_1(R_2C_2s + 1) + R_2(R_1C_1s + 1)} = \frac{C_1}{C_1 + C_2} \left[ \frac{s + \frac{1}{R_1C_1}}{s + \frac{1}{R_1/R_2(C_1 + C_2)}} \right]
\]

\[
\phi = \tan^{-1}\left(\frac{1}{R_1C_1}\right) - \tan^{-1}\left(\frac{1}{(R_1/R_2)(C_1 + C_2)}\right) = 0
\]

\[0 = R_1C_1 - (R_1/R_2)(C_1 + C_2)\]

\[R_1C_1 = (R_1/R_2)(C_1 + C_2)\]

\[
\frac{C_1}{C_1 + C_2} = \frac{R_2}{R_1 + R_2} = \frac{1}{10}
\]

\[
\begin{array}{c|c|c}
C_1 & C_2 & 9C \\
\hline
C_1 & C_2 & 9C
\end{array}
\]
Exercise 10.2

From the ILAB Server: \( f_o = 1.548 \text{ kHz} \)

\[ \omega_0 = 2\pi f_o = 9852 \text{ rad/s} \]

This is unit of \( \phi = 0 \) and near lowest value of notch.

Since \( \omega_0 = \frac{1}{\sqrt{LC}} \)

\[ C = 0.21 \mu F \]

\[ H_n(s) = \frac{V_0(s)}{V_{in}(s)} = \frac{s^2 + \frac{1}{LC}}{s^2 + 2\pi \times 5 + \omega_0^2} \]

\[ 2\pi \omega_0 = \frac{L}{\pi} \]

\[ |H_n(j\omega)| = \frac{1 - \omega^2 LC}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}} \]

From plot \( |H_n(j\omega)| \) at \( \omega = 2\pi \times 1.195 \text{ kHz} \)

\[ |H_n(j\omega)| = 12.49 \text{ m} \]

\[ R = 10 \text{ k}\Omega \]
**Problem 10.1:** This problem explores the Thévenin and Norton equivalence of networks operating in sinusoidal steady state. All networks considered here are comprised of linear resistors, capacitors and inductors, and voltage and current sources all operating at the same frequency \( \omega \). Therefore, all branch currents and voltages operate at the frequency \( \omega \).

(A) Determine the relations between \( V_T, Z_T, I_N \) and \( Z_N \) which must exist for the \( i-v \) relations at the terminals of Networks \#1 and \#2 to be identical when operating in sinusoidal steady state.

(B) Review the arguments for networks involving only resistors and sources, and then briefly explain why Networks \#1 and \#2 may serve as the Thévenin and Norton equivalents, respectively, of an arbitrary network operating in sinusoidal steady state.

(C) Determine \( V_T \) and \( Z_T \) in the Thévenin equivalent of Network \#3.

(D) Suppose \( Z_T \) in Part C is implemented with Networks \#4 and \#5. Determine \( R_T \) and \( C_T \) in Network \#4, and \( R_T \) and \( L_T \) in Network \#5, in terms of \( R, L, C \) and \( \omega \). Under what circumstances is Network \#4 preferred over Network \#5, and vice versa?

![Network Diagrams](image)

**Answer:** Since we know that \( Z_T \) and \( Z_N \) are complex, we can rewrite them in polar form:

\[
Z_T = |Z_T|e^{j\phi_Z_T} \\
Z_N = |Z_N|e^{j\phi_Z_N}
\]

(A) We can determine the relations between \( V_T, Z_T, I_N \) and \( Z_N \) by applying a complex test voltage, \( v(t) = \tilde{v}e^{j\omega t} \), into the ports of both \#1 and \#2, and determining the complex current \( i(t) = ie^{j\omega t} \) flowing into the positive terminal. For \#1 we get:

\[
i_e^{j\omega t} = \frac{\tilde{v}e^{j\omega t} - V_Te^{j\omega t}}{|Z_T|e^{j\phi_Z_T}}
\]

Dividing both sides by \( e^{j\omega t} \), we get:

\[
i = \frac{\tilde{v}}{|Z_T|e^{j\phi_Z_T}} - \frac{V_T}{|Z_T|e^{j\phi_Z_T}}
\]

For \#2 we get:

\[
i_e^{j\omega t} = \frac{\tilde{v}e^{j\omega t} - I_Ne^{j\omega t}}{|Z_N|e^{j\phi_Z_N}}
\]
\[ i = \frac{|Z_N|e^{j\phi_{Z_N}}}{|Z_T|e^{j\phi_{Z_T}}} = i_N \]

By observing that the coefficient in front of both \( \dot{v} \) terms must be equal, we obtain a relationship between \( Z_N \) and \( Z_T \):

\[ Z_T = |Z_T|e^{j\phi_{Z_T}} = |Z_N|e^{j\phi_{Z_N}} = Z_N \]

or simply

\[ Z_T = Z_N \]

(D) Suppose \( Z_T \) in Part C is implemented with Networks #4 and #5. Determine \( R_T \) and \( C_T \) in Network #4, and \( R_T \) and \( L_T \) in Network #5, in terms of \( R, L, C \) and \( \omega \). Under what circumstances is Network #4 preferred over Network #5, and vice versa?

Answer: Since we know that \( Z_T \) and \( Z_N \) are complex, we can rewrite them in polar form:

\[ Z_T = |Z_T|e^{j\phi_{Z_T}} \]
\[ Z_N = |Z_N|e^{j\phi_{Z_N}} \]

(A) We can determine the relations between \( V_T, Z_T, I_N \) and \( V_N \) by applying a complex test voltage, \( v(t) = v_0e^{j\omega t} \), into the ports of both #1 and #2, and determining the complex current \( i(t) = i_0e^{j\omega t} \) flowing into the positive terminal. For #1 we get:

\[ \dot{v}_0e^{j\omega t} = \frac{\dot{v}e^{j\omega t} - V_T e^{j\omega t}}{|Z_T|e^{j\phi_{Z_T}}} \]

Dividing both sides by \( e^{j\omega t} \), we get:

\[ \dot{i} = \frac{\dot{v}}{|Z_T|e^{j\phi_{Z_T}}} = \frac{V_T}{|Z_T|e^{j\phi_{Z_T}}} \]

For #2 we get:

\[ i_0e^{j\omega t} = \frac{\dot{v}e^{j\omega t} - I_N e^{j\omega t}}{|Z_N|e^{j\phi_{Z_N}}} \]

which gives us:

\[ i = \frac{\dot{v}}{|Z_N|e^{j\phi_{Z_N}}} = I_N \]

By observing that the coefficient in front of both \( \dot{v} \) terms must be equal, we obtain a relationship between \( Z_N \) and \( Z_T \):

\[ Z_T = |Z_T|e^{j\phi_{Z_T}} = |Z_N|e^{j\phi_{Z_N}} = Z_N \]
since $|Z_T| = |Z_N|$ and $\phi_{Z_T} = \phi_{Z_N}$. Also, the right-hand most terms must be equivalent for the $i-v$ relationship to be the same:

$$I_N = \frac{V_T}{|Z_T|e^{j\phi_{Z_T}}} = \frac{V_T}{Z_T}$$

This result could be shown to hold for real sinusoidal inputs by taking the real part of the resulting equations, which would result in the same analysis. We can note that Thevenin and Norton equivalences can be used when dealing with impedances.

(B) We note from out above analysis that the two circuits can be used as Thevenin and Norton equivalents when the inputs are sinusoidal because we see that for any arbitrary sinusoidal voltage of characteristic frequency $\omega$ and phase $\phi_v$, there is a direct mapping to a specific sinusoidal current of the same frequency with a specific phase. If the same voltage were applied to both output terminals of the two circuits, then the same current (including phase and magnitude) would be observed. Thus, if either circuit were hooked up to an arbitrary network, they would function exactly the same and thus be electronically interchangeable (i.e. equivalent).

(C) Using our impedance transformations, we can determine a Thevenin equivalent circuit for #3. First, the Thevenin Impedance, $Z_T$, can be found by shorting out (setting to 0) the voltage source on the left-hand side and finding the total equivalent resistance as seen from the two ports:

$$Z_T = R||\left(\frac{1}{j\omega C}\right) = \frac{R - \omega^2 RLC}{1 - \omega^2 LC + j\omega RC}$$

The Thevenin voltage can be found by assuming no current flows into the positive terminal ($i = 0$) and measuring the terminal voltage. This becomes a voltage divider:

$$V_T = V \frac{R}{R + j\omega L + \frac{1}{j\omega C}}$$

Note that, in particular, $V_T$ is complex, and therefore has a phase shift.
a) \[ Z_T = Z_0 \parallel (Z_R + Z_L) \]
\[ = \frac{1}{j\omega C} \left( R + j\omega L \right) \]
\[ \frac{1}{j\omega C + R + j\omega L} \]
\[ Z_T = \frac{|R+j\omega L|}{(1-\frac{1}{\omega LC}) + j\omega RC} \]

b) For Network #4:
\[ Z_y = R_T + \frac{1}{j\omega C_T} = R_T - j\frac{1}{\omega C_T} \]
Heightened negative reactance.

For Network #5:
\[ Z_S = R_T + j\omega L_T \]
Heightened positive reactance.

If we wish for #4 to be equivalent to \( Z_T \) of Network #3, real parts must be equal and imaginary parts must be equal.
\[ z_T#3 = \frac{R + j \omega L}{\left(1 - \omega^2 LC\right) - j \omega R C} \times \frac{\left(1 - \omega^2 LC\right) - j \omega R C}{\left(1 - \omega^2 LC\right) + j \omega R C} \]

\[ = \frac{R \left(1 - \omega^2 LC\right) + \omega^2 RLC + j \left(\omega \left(1 - \omega^2 LC\right) - \omega R^2 C\right)}{\left(1 - \omega^2 LC\right)^2 + (\omega R C)^2} \]

\[ = R + j \omega \left[ \frac{L \left\{1 - \omega^2 LC\right\} - R^2 C}{\left(1 - \omega^2 LC\right)^2 + (\omega R C)^2} \right] \]

**#4**

\[ R_T = \frac{R}{\left(1 - \omega^2 LC\right)^2 + (\omega R C)^2} \]

\[ C_T = \frac{(1 - \omega^2 LC)^2 + (\omega R C)^2}{L \left\{1 - \omega^2 LC\right\} - R^2 C} \]

**#5**

\[ R_T = \frac{R}{\left(1 - \omega^2 LC\right)^2 + (\omega R C)^2} \]

\[ L_T = \frac{L \left\{1 - \omega^2 LC\right\} - R^2 C}{\left(1 - \omega^2 LC\right)^2 + (\omega R C)^2} \]
**Problem 10.1 continued**

Depending on whether \( w \) is less than or greater than \( \frac{1}{\sqrt{LC}} \), \( L \) or \( C \) will be a positive value.

Since capacitors and inductors always have positive, real values for \( C + L \), we would implement the network that would give us positive values for \( L \) or \( C \).

For \( w \leq \frac{1}{\sqrt{LC}} \) \( \Rightarrow \) network \# 5.

For \( w > \frac{1}{\sqrt{LC}} \) \( \Rightarrow \) network \# 4.
Problem 10.2: This problem examines the very simple tuner for an AM radio shown below. Here, the tuner is the parallel inductor and capacitor. The injection of radio signals into the tuner by the antenna is modeled by a current source, while the Norton resistance of the antenna in parallel with the remainder of the radio is modeled by a resistor. (You will learn about antenna modeling in 6.014.) The AM radio band extends from 540 kHz through 1600 kHz. The information transmitted by each radio station is constrained to be within ±5 kHz of its center frequency. (You will learn about AM radio transmission in 6.003.) To prevent frequency overlap of neighboring stations, the center frequency of each station is constrained to be a multiple of 10 kHz. Therefore, the purpose of the tuner is to pass all frequencies within 5 kHz of the center frequency of the selected station, while attenuating all other frequencies.

(A) Assume that \( I(t) = I \cos(\omega t) \). Find \( v(t) \) where \( v(t) = V \cos(\omega t + \phi) \), and both \( V \) and \( \phi \) are functions of \( \omega \). Note that \( v(t) \) is the output of the tuner, namely the signal that is passed on to the remainder of the radio.

(B) For a given combination of \( I \), \( C \), \( L \) and \( R \), at what frequency is \( V \) maximized?

(C) Assume that \( L = 365 \mu H \). Over what range of capacitance must \( C \) vary so that the frequency of maximum \( V/I \) may be tuned over the entire AM band. Note that tuning the frequency of maximum \( V/I \) to the center frequency of a particular station tunes in that station.

(D) As a compromise between passing all frequencies within 5 kHz of a center frequency and rejecting all frequencies outside that band, let the design of \( R \) be such that \( V(1 \text{ MHz} \pm 5 \text{ kHz})/V(1 \text{ MHz}) \approx 0.25 \) when the tuner is tuned to 1 MHz. Given this design criterion, determine \( R \).

(E) Given your design for \( R \), determine \( V(1 \text{ MHz} \pm 10\text{kHz})/V(1 \text{ MHz}) \). Also, determine \( Q \) for the tuner and its load resistor when the tuner is tuned to 1 MHz.

(F) Suppose the tuner is tuned to another station and then quickly tuned to the station broadcasting at 1 MHz. Approximately how long will it take for \( v(t) \) to depend primarily on the signal from the station broadcasting at 1 MHz. Assume that both stations broadcast signals of equal strength. Hint: consider the time-domain interpretation of \( Q \).
**Answer:**

(A) Using the impedance transformations, we can write

\[ \tilde{V} = I \cdot Z_{eq} = \frac{I}{Y_{eq}} \]

Noting that admittances in parallel add, we find

\[ \tilde{V} = \frac{I}{\frac{L}{j\omega} + j\omega C + \frac{I}{R}} \]

\[ = \frac{j\omega L}{j\omega^2 LC + j\omega \frac{I}{R} + 1} \cdot I \]

\[ = \frac{j\omega L}{1 - \omega^2 LC + j\omega \frac{I}{R}} \cdot I = V e^{j\phi} \]

Taking the magnitude and the phase of this complex expression for \( \tilde{V} \), we find

\[ V = I \cdot \frac{\omega L}{\sqrt{(1 - \omega^2 LC)^2 + \left(\omega \frac{L}{R}\right)^2}} \]

\[ \phi = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega \frac{L}{R}}{1 - \omega^2 LC}\right) = \tan^{-1}\left(\frac{1 - \omega^2 LC}{\omega \frac{L}{R}}\right) \]

\[ v(t) = V \cos(\omega t + \phi) \]

(B) For the purposes of this section, let

\[ X = (1 - \omega^2 LC)^2 + \left(\frac{\omega L}{R}\right)^2 \]

To find \( \omega \) when \( V \) is maximized, we have to find \( \omega \) such that \( \frac{dV}{d\omega} = 0 \). Looking at the numerator of the messy derivative, we see that

\[ 0 = IL\sqrt{X} - \frac{I\omega L}{2\sqrt{X}} \left[ 2(1 - \omega^2 LC)^2(-2\omega LC) + 2\omega \left(\frac{L}{R}\right)^2 \right] \]

\[ 0 = X - \omega \left[ -2\omega LC(1 - \omega^2 LC) + \omega \left(\frac{L}{R}\right)^2 \right] \]

\[ 0 = (1 - \omega^2 LC)^2 + \omega^2 \frac{L^2}{R} + 2\omega^2 LC - 2\omega^4 L^2 C^2 - \omega^2 \frac{L^2}{R} \]
Solving this polynomial for \( \omega \) we find

\[
\omega_{\text{max}(\nu)} = \omega_N = \frac{1}{\sqrt{LC}}
\]

(C) To operate over the entire AM band, \( 540 \text{ kHz} \leq \frac{\omega_k}{2\pi} \leq 1600 \text{ kHz} \), we can write

\[
540 \times 10^3 \leq \frac{1}{2\pi\sqrt{LC}} \leq 1600 \times 10^3
\]

Rearranging terms to find \( C \) yields

\[
\frac{1}{L(2\pi(540 \times 10^3))^2} \geq C \geq \frac{1}{L(2\pi(1600 \times 10^3))^2}
\]

Substituting in \( L = 365 \mu\text{H} \) gives

\[
238 \text{ pF} \geq C \geq 27.1 \text{ pF}
\]

(D) First, find the amplitude at the center frequency 1 MHz using Parts (A) and (B).

\[
\phi_C = \phi_N = \frac{1}{\sqrt{LC}}, \quad |V| = IR
\]

The frequency 5 kHz away can be represented as a fraction \( \gamma \) of the center frequency.

\[
\omega = \gamma \omega_N = \frac{\gamma}{\sqrt{LC}}
\]

\[
V(\gamma \omega_N) = \frac{I \gamma \frac{L}{\sqrt{LC}}}{\sqrt{(1 - \gamma^2)^2 + \left(\frac{2L}{\sqrt{LC}}\right)^2}} = \frac{IL}{\sqrt{LC} \left(\frac{1}{\gamma} - \gamma\right)^2 + \left(\frac{1}{R}\right)^2}
\]

\[
\frac{V(\gamma \omega_N)}{V(\omega_N)} = \frac{L}{\sqrt{R^2LC \left(\frac{1}{\gamma} - \gamma\right)^2 + L^2}} \equiv \beta
\]

\[
\frac{L^2}{\beta^2} = R^2LC \left(\frac{1}{\gamma} - \gamma\right)^2 + L^2
\]

\[
\frac{L(1 - \beta^2)}{\beta^2} = R^2C \left(\frac{1}{\gamma} - \gamma\right)^2
\]

\[
\Rightarrow R = \frac{\frac{L(1 - \beta^2)}{\beta^2}}{\sqrt{\beta^2C \left(\frac{1}{\gamma} - \gamma\right)^2}}
\]

Substituting \( L = 365 \mu\text{H}, \beta = .25, C = \frac{1}{L(2\pi \times 10^3)} = 69.4 \text{ pF}, \) and \( \gamma = .995 \) gives

\[
R = 886 \text{ k}\Omega
\]

using \( \gamma = 1.005 \) gives

\[
R = 899 \text{ k}\Omega
\]

We will choose the case where \( \frac{V(\gamma \omega_N)}{V(\omega_N)} > .25 \). Since \( \omega = .995 \text{ MHz} \) is a tighter constraint on \( R \):

\[
R = 886 \text{ k}\Omega
\]
(E) \( V(1 \text{ MHz} \pm 10 \text{ kHz}) \) is given by

\[
\frac{V(\gamma \omega_N)}{V(\omega_N)} = \frac{L}{\sqrt{R^2 L C \left( \frac{1}{\gamma} - \gamma \right)^2 + L^2}} \quad \text{where} \quad \gamma = .99
\]

Therefore

\[
\frac{V(0.99 \omega_N)}{V(\omega_N)} = .128
\]

Finding \( Q \) is quick:

\[
Q = \frac{\omega_N}{2\alpha} = \frac{RC}{\sqrt{LC}} = R\sqrt{\frac{C}{L}} = 386
\]

(F) Our circuit will have a transient that will decay away as \( e^{-\alpha t} \) as soon as we switch stations. The time constant is \( \frac{1}{\alpha} = 2RC = 123 \mu s \).

There are many reasonable engineering approximations for when a transient has mostly died away. Note that after three time constants, the transient has decayed to \( e^{-3} \approx 0.05 \) = 5% of its initial value. \( 3 \cdot 2RC = 369 \mu s \). You could also have used a different number of time constants as your reference.

**Problem 10.3:** This problem studies capacitive coupling as used by the amplifier shown below. The amplifier employs both input and output capacitive coupling. That is, the input voltage source \( v_{IN} \) is coupled to the amplifier through the capacitor \( C_1 \), and the load resistor \( R_L \) is coupled to the amplifier through the capacitor \( C_O \).
Answer:

(A) Under the constant bias condition, the capacitors all behave like open circuits. Therefore, \( V_{GS} \) is set by the voltage divider of the two bias resistors \( R_{G1} \) and \( R_{G2} \) and the bias source \( V_S \):

\[
V_{GS} = \frac{R_{G2}}{R_{G1} + R_{G2}} \cdot V_S
\]

(B) No. Because the voltage divider biases the gate to a DC value, it is not necessary to also have a bias voltage on the input. Further, because the capacitor \( C_1 \) is an open circuit for biasing purposes, an input bias is also not useful.

(C) Small Signal Model:
where \( R_{bias} = \frac{R_{G1}R_{G2}}{R_{G1} + R_{G2}} \)

(D) We can find the complex transfer function from \( v_{in} \) to \( v_{gs} \) and the complex transfer function from \( v_{gs} \) to \( v_{out} \) and then multiple them to find the total complex transfer function from \( v_{in} \) to \( v_{out} \).

\[
\frac{V_{out}}{V_{in}} = \left| \frac{\vec{v}_{out}}{\vec{v}_{in}} \right| = V_{in} \cdot \frac{|\vec{v}_{gs}|}{|\vec{v}_{in}|} \cdot \frac{|\vec{v}_{out}|}{|\vec{v}_{gs}|}
\]

\[
\phi = \angle \left( \frac{\vec{v}_{gs}}{\vec{v}_{in}} \right) + \angle \left( \frac{\vec{v}_{out}}{\vec{v}_{gs}} \right)
\]

\[
\frac{\vec{v}_{gs}}{\vec{v}_{in}} = \frac{1}{R_{bias} + j\omega C_{GS} + R_{bias} C_1 + j\omega C_{GS}}
\]

\[
\frac{\vec{v}_{gs}}{\vec{v}_{in}} = \frac{R_{bias}}{(R_{bias} + C_1) + j\omega C_{GS} R_{bias} C_1}
\]

\[
\frac{\vec{v}_{out}}{\vec{v}_{gs}} = \frac{(-)R_{L}g_{m}R_{D}}{(R_{L} + R_{D}) + \frac{1}{j\omega C_{D}}}
\]
\[
\frac{\ddot{v}_{\text{out}}}{\ddot{v}_{\text{gs}}} = \frac{-j\omega RLg_mR_DC_O}{1 + j\omega C_O(R_L + R_D)} 
\]

Now find the magnitude and phase of the two transfer functions to determine \(V_{\text{out}}\) and \(\phi\).

\[
V_{\text{out}} = V_{\text{in}} \left| \frac{\ddot{v}_{\text{out}}}{\ddot{v}_{\text{in}}} \right| = V_{\text{in}} \left| \frac{\ddot{v}_{\text{gs}}}{\ddot{v}_{\text{in}}} \right| = V_{\text{in}} \frac{\omega g_mRLRD\text{Bias}C_O}{\sqrt{[(R_{\text{bias}} + C_I)^2 + (\omega C_{GS}R_{\text{bias}}C_I)^2] \cdot [1 + (\omega C_O(R_L + R_D)^2]}}
\]

\[
\phi = -\tan^{-1}\left(\frac{\omega C_{GS}R_{\text{bias}}C_I}{R_{\text{bias}} + C_I}\right) - \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega C_O(R_L + R_D)}{1}\right)
\]