Ex 2.1

The easiest way to solve this problem is to exploit the power relation so that the value of the resistors is the only variable in the equation.

For this system, the voltage across all resistors is the same: $V = I(1k||2k||5k)$

- $P = VI = V^2 / R$ for each resistor
  - $P_{1k} = V^2 / 1k$
  - $P_{2k} = V^2 / 2k$
  - $P_{5k} = V^2 / 5k$

Since the 1k resistor draws the most current, it burns the hottest.

For this system, the current through all the resistors is the same: $I = \frac{V}{1k + 2k + 5k}$

- $P = VI = I^2 R$ for each resistor
  - $P_{1k} = I^2 (1k)$
  - $P_{2k} = I^2 (2k)$
  - $P_{5k} = I^2 (5k)$

Since the 5k resistor provides the largest voltage drop, it consumes the most power.
This network does not generalize as neatly as the previous two. We will need to do some algebra to solve it.

\[
P_{1K} = I^2(1K)
\]

Since all current flows through \(R_{1K}\).

\[
P_{2K} = \left(\frac{I(5K)}{2K+5K}\right)^2 \cdot 2K
\]

where \(\frac{I(5K)}{2K+5K}\) is the current divider relation for \(R_{2K}\).

\[
P_{5K} = \left(\frac{I(2K)}{2K+5K}\right)^2 \cdot 5K
\]

\[
P_{7K} = \left(\frac{I}{4K}\right)I^2 \cdot 5K
\]

So the 2K resistor will just barely dissipate the most power.

**Ex 2.2**

You should feel confident enough to build these matrices by inspection. Where row 1 column 1 is the sum of all conductances connected to \(e_1\). Row 1 column 2 is the negative conductance between \(e_2\) and \(e_1\), etc.

For node currents out of each node, the matrix pair looks like:

\[
\begin{bmatrix}
G_1 + G_2 + G_3 & -G_3 & 0 \\
-G_3 & G_3 + G_{14} - G_{14} & -G_4 \\
0 & -G_4 & G_6 + G_{16} + G_6
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
= 
\begin{bmatrix}
G_1 \, V \\
I \\
G_{16} \, V
\end{bmatrix}
\]
Where Row 1 of S is the sum of the currents going into the node e1 (note sign change from analogous terms in G).

To verify this matrix equation, let's write the node equations for current coming out of the nodes.

\[ e_1 : 0 = (e_1 - V)G_1 + e_1 G_2 + (e_1 - e_2)G_3 \]
\[ e_2 : 0 = (e_2 - e_1)G_3 - I \cdot (e_2 - e_3)G_4 \]
\[ e_3 : 0 = (e_3 - e_2)G_4 + e_3 G_5 + (e_3 - V)G_6 \]

Grouping like terms and moving source-driven currents to other side:

\[ e_1 : (e_1)(G_1 + G_2 + G_3) - e_2 (G_3) - e_3 (0) = G_1 V \]
\[ e_2 : (e_1)(-G_3) + e_2 (G_3 + G_4) + e_3 (-G_4) = I \]
\[ e_3 : e_1 (0) + e_2 (-G_4) + e_3 (G_4 + G_5 + G_6) = G_6 V \]

From which the matrices flesh out as before.
Problem 2.1

A) Superposition

First, remove current source

\[ e_1 = \frac{V \cdot R_z}{R_1 + R_2} \]

\[ e_2 = \frac{(e_1)(R_4)}{R_3 + R_4} \]

Next remove voltage source

\[ e_2 = I \left( \frac{R_4}{R_3 + \left[ \frac{R_2}{R_1} \right]} \right) = I \left( \frac{R_4}{R_3 + \left[ \frac{R_2}{R_1} \right]} \right) \]
\[ e_1 : \text{Current Divider + Device Laws} \]

\[ i_3 = I \left[ \frac{R_4}{R_4 + R_3 + \frac{R_2}{R_1}} \right] = I \left[ \frac{i_4}{R_4 + \frac{R_5 + \frac{R_2}{R_1}}{R_4 + \frac{R_2}{R_1}}} \right] \]

\[ e_1 = (i_3) \left( \frac{R_2}{R_1} \right) \]

\[ = \left[ \frac{R_4 \cdot \left( \frac{R_1 R_2}{R_1 + R_2} \right) I}{R_4 + R_3 + \frac{R_2}{R_1 + R_2}} \right] \]

\text{Now Combine Terms}

\[ e_1 = \frac{V \left[ \frac{R_2 (R_5 + R_4)}{R_2 + R_5 + R_4} \right]}{R_2 + R_5 + R_4} + \frac{I \cdot R_4 \cdot \left( \frac{R_2}{R_1 + R_2} \right)}{R_4 + R_3 + \frac{R_2}{R_1 + R_2}} \]

\[ e_2 = VR_4 \left[ \frac{R_2 (R_5 + R_4)}{R_2 + R_5 + R_4} \right] + \frac{I \cdot R_4 \left[ \frac{R_5 + R_2}{R_1 + R_2} \right]}{R_4 + R_3 + \frac{R_2}{R_1 + R_2}} \]

\text{B) Node Equations}

\[ \begin{align*}
V &= e_1 \\
&= (e_3 - V) G_1 + e_1 G_2 + (e_1 - e_2) G_5 = 0 \\
&= (e_2 - e_1) G_5 - I + e_2 G_{4a} = 0
\end{align*} \]
Solving via substitution

\[ e_2 (g_3 + g_4) = e_1 g_5 + I \]
\[ e_2 = \frac{e_1 g_5 + I}{g_3 + g_4} \]

\[ (e_1 - V) g_1 + e_1 g_2 + e_1 g_3 - \left( \frac{e_1 g_5 + I}{g_3 + g_4} \right) g_5 = 0 \]

\[ e_1 \left( g_1 + g_2 + g_3 - \frac{g_5^2}{g_3 + g_4} \right) = Vg_1 + \frac{Ig_5}{g_3 + g_4} \]

\[ e_1 = \frac{Vg_1 + Ig_5}{g_3 + g_4} \]
\[ g_1 + g_2 + g_3 - \frac{g_5^2}{g_3 + g_4} \]

\[ e_2 = \left( \frac{Vg_1 + Ig_5}{g_3 + g_4} \right) \left( g_3 + I \right) \]
\[ = \frac{e_1 (g_5)}{g_3 + g_4} + \frac{I}{g_3 + g_4} \]
c) The easiest way to do this is to verify that the $V$ term and the $I$ term gives the same result for each node.

### Node $V$ term

\[
\begin{align*}
V_{R_1} & = \frac{V_{R_1}}{V_{R_1} + V_{R_2} + V_{R_3} - \frac{V_{R_3}^2}{V_{R_3} + V_{R_4}}} \\
& = \frac{V_{R_1}}{V_{R_1} + V_{R_2} + V_{R_3}} - \frac{V_{R_3}^2}{V_{R_3} + V_{R_4}}
\end{align*}
\]

\[V(R_2(R_4+R_3)) = \frac{V_{R_2}^2}{R_2R_4 + R_2(R_3 + R_4) + R_1R_4 + R_2R_3 + R_2R_4}
\]

\[V_{R_2}^2 = \frac{V_{R_2}^2}{R_2R_4 + R_2R_3 + R_2R_4 + R_1R_4 + R_2R_3 + R_2R_4}
\]

### Node $I$ term

\[I_{R_2} = \frac{R_2(R_3 + R_4)}{R_2 + R_3 + R_4} \quad I_{R_2} = \frac{I_{R_2}}{I_{R_2} + I_{R_3} + I_{R_4}}
\]

\[I_{R_2} = \frac{R_2R_3 + R_2R_4}{R_2R_3 + R_2R_4 + R_1R_3 + R_1R_4}
\]

### Node $V$ term

\[V_{R_2} = \frac{V_{R_2}}{V_{R_2} + V_{R_3} - \frac{V_{R_3}^2}{V_{R_3} + V_{R_4}}} \\
& = \frac{V_{R_2}}{V_{R_2} + V_{R_3}} - \frac{V_{R_3}^2}{V_{R_3} + V_{R_4}}
\]

\[V_{R_2} = \frac{V_{R_2}}{V_{R_2} + V_{R_3}} - \frac{V_{R_3}^2}{V_{R_3} + V_{R_4}}
\]

### Node $I$ term

\[I_{R_2} = \frac{R_2(R_3 + R_4)}{R_2 + R_3 + R_4} \quad I_{R_2} = \frac{I_{R_2}}{I_{R_2} + I_{R_3} + I_{R_4}}
\]

\[I_{R_2} = \frac{R_2R_3 + R_2R_4}{R_2R_3 + R_2R_4 + R_1R_3 + R_1R_4}
\]

\[I_{R_2} = \frac{I_{R_2}}{I_{R_2} + I_{R_3} + I_{R_4}}
\]
Problem 2.2

A. To find Norton/Thévenin Equivalents, we need \( V_{oc}, I_{sc}, R_{th} \)

For both networks we know:

\[ V_{oc} = \text{Voltage when no current flows across terminals} \]
\[ = V \text{- intercept} \]

\[ I_{sc} = \text{Current when there is no voltage difference across terminals. This will be negative because the graph is written such that } i \text{ flows into the positive terminal.} \]

\[ R_{th} = \frac{V_{oc}}{I_{sc}} = \text{slope of } VI \text{ characteristic for network} \]

\[ N_1: \]
\[ V_{oc} = -V_i \]
\[ I_{sc} = -I_i \]
\[ R_{th} = \frac{-V_i}{I_i} \]

\[ N_2: \]
\[ V_{oc} = V_z \]
\[ I_{sc} = +I_z \]
\[ R_{th} = \frac{+V_z}{I_z} \]

So our two possible circuits are

Thévenized

\[ -V_i \quad \frac{V_i}{R} \quad \frac{+V_z}{R} \]

Nortonized

\[ -I_i \quad \frac{V_i}{R} \quad \frac{+V_z}{R} \quad +I_z \]

B. Using superposition with the Nortonized Circuit
\[ i_1 = \frac{-I_1 \left( \frac{V_{I_1}}{V_{I_1}} \right)}{V_{I_1} + R} \]
\[ = \frac{I_1 \left( \frac{V_{I_1}}{V_{I_1}} \right)}{V_{I_1} + \frac{R}{V_{I_2} / R}} \]

\[ i_2 = \frac{-I_2 \left( \frac{V_{I_2}}{V_{I_2}} \right)}{V_{I_2} + R} \]
\[ = \frac{I_2 \left( \frac{V_{I_2}}{V_{I_2}} \right)}{V_{I_2} + \frac{R}{V_{I_1} / R}} \]

\[ = \frac{V_{I_1} R}{V_{I_1} + R} + \frac{V_{I_2}}{I_2} \]

by current divider

\[ i_1' = \frac{-I_1 \left( \frac{V_{I_1}}{V_{I_1}} \right)}{V_{I_1} + R} \]
\[ = \frac{I_1 \left( \frac{V_{I_1}}{V_{I_1}} \right)}{V_{I_1} + \frac{R}{V_{I_2} / R}} \]

\[ i_2' = \frac{-I_2 \left( \frac{V_{I_2}}{V_{I_2}} \right)}{V_{I_2} + R} \]
\[ = \frac{I_2 \left( \frac{V_{I_2}}{V_{I_2}} \right)}{V_{I_2} + \frac{R}{V_{I_1} / R}} \]

\[ = \frac{V_{I_1} R}{V_{I_1} + R} + \frac{V_{I_2}}{I_2} \]

Analogously:

\[ i_1 = I_1 \left( \frac{\frac{V}{I_1}}{\frac{V}{I_1}} \right) \]
\[ = \frac{I_1 \left( \frac{V}{I_1} \right)}{V_{I_1} / R + V_{I_2} / I_2} \]

\[ i_2 = I_2 \left( \frac{\frac{V}{I_2}}{\frac{V}{I_2}} \right) \]
\[ = \frac{I_2 \left( \frac{V}{I_2} \right)}{V_{I_2} / R + V_{I_1} / I_1} \]

Combining:

\[ i_1 = \frac{-I_1 V_{I_1}}{V_{I_1} + \frac{R V_{I_2}}{R + V_{I_2}}} + \frac{I_2 \left( \frac{V_{I_2} R}{V_{I_2} / R + V_{I_2}} \right)}{V_{I_2} / R + V_{I_2}} \]

\[ i_2 = \frac{-I_1 V_{I_1}}{V_{I_1} + R} + \frac{I_2 \left( \frac{V_{I_2} R}{V_{I_2} / R + V_{I_2}} \right)}{V_{I_2} / R + V_{I_2}} \]
Problem 2.3

A) Again, we will need $V_{oc}, I_{sc},$ and $R_{th}$ to solve this problem.

\[ R_{th} = R_{1} \parallel R_{2} + R_{5} \parallel R_{4} \]

\[ R_{th} = \frac{R_{1} R_{2}}{R_{1} + R_{2}} + \frac{R_{5} R_{4}}{R_{5} + R_{4}} \]

$V_{oc}$:

\[ V_{oc} = V_{+} - V_{-} \]

\[ V_{+} = \frac{V R_{2}}{R_{1} + R_{2}} \]

\[ V_{-} = \frac{V R_{4}}{R_{5} + R_{4}} \]

$I_{sc}$:

By KCL, we know $I_{sc} = I_{2} - I_{1}$

\[ I = \frac{V}{R_{2} \parallel R_{4} + R_{11} R_{5}} \]

\[ I_{2} = \frac{I R_{4}}{R_{2} + R_{4}} \]

\[ I_{1} = \frac{I R_{5}}{R_{1} + R_{3}} \]
Defining current into eq circuit as negative, 

Note that \( I_{sc} = -\frac{V_{oc}}{R_{th}} = V \left[ \frac{R_2 + R_4}{R_1 R_2 + R_3 + R_4} \right] \)

\[ = V \left[ \frac{-R_2 R_3 + R_2 R_4 + R_1 R_4 - B Z R_4}{R_1 R_2 R_3 + R_1 R_2 R_4 + R_3 R_4 R_1 + R_2 R_3 R_4} \right] \]

\[ I_{sc} = V \left[ \frac{R_1 R_4 - B Z R_3}{R_1 R_2 R_3 + R_1 R_2 R_4 + R_3 R_4 R_1 + R_2 R_3 R_4} \right] \]

these are equivalent \( \Rightarrow \)

\[ I_{sc} = V \cdot \left( \frac{1}{R_1 R_4 + R_2 + R_3} \right) \left[ \frac{R_4}{R_2 + R_4} - \frac{R_3}{R_1 + R_3} \right] \]

\[ = V \cdot \left[ \frac{R_1 R_4 - R_2 R_3}{R_1 R_2 R_3 + R_1 R_2 R_4 + R_3 R_4 R_1 + R_2 R_3 R_4} \right] \]

\[ I_{sc} = \frac{R_4 R_1 - R_2 R_3}{R_1 R_2 R_3 + R_1 R_2 R_4 + R_3 R_4 R_1 + R_2 R_3 R_4} \]

\[ \text{Thevenin Eq} \]

\[ \text{Norton Eq} \]

\[ \text{where } V_{oc}, I_{sc}, R_{th} \text{ are as above.} \]
B) We need to exploit superposition repeatedly to simplify this circuit.

Let's thenenize the dashed part first

\[ V_{oc} = \frac{V_3 (2R)}{2R + 2R} = \frac{V_3}{2} \]

\[ R_{th} = \frac{2R}{2R + 2R} = R \]

Now we have

Thenenizing again

\[ V_{oc} = \frac{V_3/2}{2R + 2R} = \frac{V_3}{4} \]

\[ R_{th} = \frac{2R}{2R + 2R} = R \]
Via superposition

\[ V_{oc} = \frac{V_0/2 \cdot (2R)}{2R + R + R} + \frac{V_2 \cdot (R + R)}{2R + R + R} = V_{3/4} + V_{3/2} \]

\[ R_{th} = \frac{2R}{R + R} = R \]

Now we have

Thevenizing again

\[ V_{oc} = \frac{[V_{3/4} + V_{3/2} \cdot 2R]}{2R + R + R} + \frac{V_1 \cdot (R + R)}{2R + R + R} = V_{3/8} + V_{3/4} + V_{3/2} \]

\[ R_{th} = \frac{2R}{R + R} = R \]
Now we are left with

\[
\begin{align*}
V_{oc} &= \frac{(V_{1/2} + V_{3/4} + V_{5/8})(2R)}{2R + R + R} = \frac{V_{1/4} + V_{3/8} + V_{4/16}}{R} \\
R_{th} &= R + R/2R = R \\
I_{sc} &= -\frac{V_{oc}}{R_{th}} = \frac{V_{1/4} + V_{3/8} + V_{4/16}}{R}
\end{align*}
\]
Problem 2.4

A) \( V_T = \frac{kT}{q} = \left( \frac{1.38 \times 10^{-23} \text{J/K}}{1.60 \times 10^{-19} \text{C}} \right) \left( 297.2 \text{K} \right) = 0.0256 \text{ V} \)
\[ = 25.6 \text{ mV} \]

B) \( i_D \approx -I_s \) to approximately 0V (see log plot)

From the theoretical relation, we would expect \( i_D \) to begin increasing as soon as \( e^{v_{ocT}} \approx 1 \) so at 0V.

For given data, \( I_s \approx +2.2 \text{ nA} \).

C) \( i_D \) begins to increase linearly, instead of exponentially, above about 0.35V (this is easier seen on a non-logarithmic graph). This is not as expected.

D) If such a resistor is placed in series with the diode, its linearity would dominate the exponential diode as soon as the diode's current reaches the level where it is equal to a resistive current.

\[ \text{Resistance} = (\text{slope of linear region})^{-1} \left( 0.032 \text{V} \right)^{-1} \]

\[ R_s \approx 311 \Omega \]
To find $A$, we need to plot the load line given by the rest of the circuit and find its intercept
with our diode data curve.

\[ V_D = A - (I_D)(R) \]
\[ I_D = \frac{A - V_D}{R} \]

Taking the intercept (next page)
we see $A \approx 0.4V$

To find $B$, we just need the slope of our linearized small-signal model... so the derivative of our diode expression evaluated at $A$!

\[ \frac{d}{dv} \left[ I_s \left( e^{\frac{V_D}{V_T}} - 1 \right) \right] = \frac{1}{V_T} I_s e^{\frac{V_D}{V_T}} \]

Plugging in the values from 2.4 a-d and evaluating at $V_D = 0.4V$

$B \approx 0.525$

Note that this value is incredibly sensitive to $A$. Guessing $A = 0.45V$ yields a $B$ of $-3.7$. Since $A$ is an approximation, your $B$ may be quite different and still be right.