Exercise 6.1

We know from 2.4(c) that when we have $V_D \gg V_T$, the expression for $I_o$ vs $V_D$ is

$$I_o \propto I_s e^{V_D/V_T}$$

Expanding this expression

$$I_o = I_d + I_D = I_s e^{V_D/V_T} + I_s e^{-V_D/V_T} \cdot V_D$$

$$I_d = I_D e^{V_D/V_T} \cdot V_D \Rightarrow this \ looks \ like \ a \ resistor \ with \ value \ \frac{V_T}{I_D}$$

$V_d = \frac{(Vin)(V_T/I_D)}{(V_T/I_D) + R} = \frac{Vin}{1 + R \frac{I_D}{V_T}}$
Exercise 6.2

Inductors combine like resistors

\[ L_{eq} = \frac{L_1 + L_2 + L_3}{L_2 + L_3} \]

Capacitors are duals of resistors and add like conductances.

\[ C_{eq} = \frac{(C_1)(C_2 + C_3)}{C_1 + C_2 + C_3} \]

\[ = \frac{C_1C_2 + C_1C_3}{C_1 + C_2 + C_3} \]
A) At $t=0^-$, the MOSFET is off and has been for a long time, so we have the circuit below. We know that at long time intervals, $i_Q$ goes to 0 and $v_{Qn}$ goes to $V_S$ because the capacitor will be fully charged.

Since the input is not a delta function, $V_Q$ will be continuous.

$$V_{Qn}(0^+) = V_S.$$

When the MOSFET switches on:

We now have a divider hooked to the capacitor.

Applying KCL to the node:

$$\frac{V_{Qn} - V_S}{R_{pu}} + \frac{V_{Qn}}{R_{on}} + i_Q = 0$$

$$\frac{V_{Qn} - V_S}{R_{pu}} + \frac{V_{Qn}}{R_{on}} + C \frac{dV_{Qn}}{dt} = 0$$

$$V_{Qn} + \frac{R_{on} R_{pu}}{R_{on} + R_{pu}} C \frac{dV_{Qn}}{dt} = \frac{V_S R_{on}}{R_{on} + R_{pu}}$$

First, find Homogeneous solution

$$V_{Qn} + \frac{R_{on} R_{pu}}{R_{on} + R_{pu}} C \frac{dV_{Qn}}{dt} = 0$$

$$V_{Qn}(t) = A e^{-t/\tau} \text{ where } \tau = \frac{C (R_{on} R_{pu})}{R_{on} + R_{pu}}$$
Now find particular solution

\[ V_{gp}(t) + \frac{R_{sw} R_{pm}}{R_{sw} + R_{pm}} \frac{dV_{gp}(t)}{de} = \frac{V_s R_{sw}}{R_{sw} + R_{pm}} \]

choose \( V_{gp}(t) = \frac{V_s R_{sw}}{R_{sw} + R_{pm}} \) and \( \frac{dV_{gp}}{de} = 0 \)

Now use initial conditions to find \( A \)

\[ V_a(0) = \frac{V_s R_{sw}}{R_{sw} + R_{pm}} + A e^0 = Vs = V_{gp} + V_{ah} \]

\[ A = Vs - \frac{V_s R_{sw}}{R_{sw} + R_{pm}} \]

Now combine,

\[ V_a(t > 0) = V_{gp} + V_{ah} = \left[ \frac{Vs \cdot e^{-t/e} + \frac{V_s R_{sw}}{R_{sw} + R_{pm}} \left( 1 - e^{-t/e} \right)}{R_{sw} + R_{pm}} \right] \]

where \( t = \frac{R_{sw} R_{pm}}{R_{sw} + R_{pm}} \)

**B)**

At \( t=0 \), \( V_a = Vs \), but after switch closes, \( V_a \) behaves according to the differential equation in \( A \).

\[ V_a(t = t_c) = V_{ol} = Vs e^{-t_c/e} + \frac{V_s R_{sw}}{R_{sw} + R_{pm}} \left( 1 - e^{-t_c/e} \right) \]

\[ \frac{V_{ol} - \frac{V_s R_{sw}}{R_{sw} + R_{pm}}}{Vs \left( 1 - \frac{R_{sw} R_{pm}}{R_{sw} + R_{pm}} \right)} e^{-t_c/e} \]

\[ t_c = -e \ln \left[ \frac{\frac{V_{ol} - \frac{V_s R_{sw}}{R_{sw} + R_{pm}}}{Vs \left( 1 - \frac{R_{sw} R_{pm}}{R_{sw} + R_{pm}} \right)}}{Vs} \right] \]
C) Assuming the MOSFET has been on for a long time we have

\[ i_G = 0, \quad V_{GS} = \frac{V_s (R_{on})}{R_{on} + R_{pu}} \]

and when we turn the switch off, our voltage will rise back to \( V_s \) according to the circuit below.

As in (A), writing KCL for the only unsourced node gives

\[ i_G + \frac{V_{GS} - V_s}{R_{pu}} = 0 \]

\[ \frac{C}{R_{pu}} \frac{dV_{GS}}{dt} + \frac{V_{GS} - V_s}{R_{pu}} = 0 \]

\[ R_{pu} C \frac{dV_{GS}}{dt} + V_{GS} = V_s \]

**Homogeneous**

\[ R_{pu} C \frac{dV_{GS}}{dt} + V_{GS} = 0 \]

\[ V_{GS} = -\frac{V_s}{C} t \]

\[ \tau = R_{pu} C \]

**Particular**

\[ R_{pu} C \frac{dV_{GS}}{dt} + V_{GS} = V_s \]

\[ V_{GS} = V_s \]

\[ \frac{dV_{GS}}{dt} = 0 \]
Combining and using initial conditions

\[ V_{a} = A e^{-\frac{t}{R pu}} + V_s \]

\[ V_{a}(0) = A + V_s = \frac{V_s (R_{pw})}{R_{pu} + R_{pw}} \]

\[ A = \frac{V_s (R_{pw})}{R_{pu} + R_{pw}} \left( \frac{1}{R_{pw} + R_{pu}} \right) = -\frac{V_s R_{pu}}{R_{pu} + R_{pw}} \]

\[ V_{a}(t) = V_s - \frac{R_{pu} V_s}{R_{pu} R_{pw}} e^{-\frac{t}{R_{pu}} \frac{1}{C}} \]

\[ \left[ -\frac{(V_{OH} - V_s) (R_{pu} + R_{pw})}{V_s R_{pu}} \right] = e^{-\frac{t}{R_{pu} C}} \]

\[ t_{R} = R \ln \left[ \frac{R_{pu} V_s}{(V_s - V_{OH}) (R_{pu} + R_{pw})} \right] \quad \text{where} \quad R = C_{pu} \]

D) Performing the same steps as B

\[ V_{a}(t_{R}) = V_{OH} = V_s - \frac{R_{pu} V_s}{R_{pu} R_{pw}} e^{-\frac{t_{R}}{R_{pu}} \frac{1}{C}} \]

E) As more gates are connected, we will be adding more and more \( C_{G} \) to our circuit. This will increase \( T \) and therefore make the rise and fall times slower. Remember that capacitances in parallel add, so there is a linear dependence. As more and more gates are connected, this \( C_{G} \) becomes the speed bottleneck in a circuit!
Problem 6.2

A) using KVL around the loop

\[ NI_1 = (-i_2)(R_{\text{load}}) \]

\[ i_2 = \frac{-NI_1}{R_{\text{load}}} \]

B) \[ V = L \frac{di}{dt} \] is the constitutive law for an inductor

\[ L \frac{di_L}{dt} = V_L = V_1(t) \]

\[ \frac{di_L}{dt} = \frac{1}{L} V_1(t) \]

\[ i_L(t) = \frac{1}{L} \int_0^t V_1(t) \, dt \]

We integrate from \(-\infty\) to \(t\) because we need to know all past history of the inductor voltage \(V_1\), but any changes after \(t\) change \(i\) after \(t\) as well.

C) Using the node method for currents out of the node connecting \(R_{\text{in}}, L,\) and \(N_iZ_i,\) we see

\[ \frac{V_1 - V_{\text{in}}}{R_{\text{in}}} + i_L + (-N_iZ_i) = 0 \]

Back-substituting from A+B

\[ \frac{V_1 - V_{\text{in}}}{R_{\text{in}}} + \frac{1}{L} \int_{-\infty}^t V_1(t) \, dt + \frac{N_i^2 V_1}{R_{\text{load}}} = 0 \]
\[ D) \frac{d}{dt}\left[ \frac{v_L - v_{in}}{R_{in}} + \frac{1}{2} \int_{-\infty}^{t} v_i(t) + \frac{N^2 v_i}{R_{load}} \right] = 0 \]

\[ = \frac{dv_i(t)}{dt} \frac{1}{R_{in}} - \frac{dv_{in}(t)}{dt} \frac{1}{R_{in}} + \frac{1}{2} v_i(t) + \frac{N^2}{R_{load}} \frac{dv_i}{dt} = 0 \]

\[ \frac{dv_i(t)}{dt} \left( \frac{1}{R_{in}} + \frac{N^2}{R_{load}} \right) + v_i(t) = \frac{L}{R_{in}} \frac{dv_{in}}{dt} \]

E) Now our equation looks like

\[ \left( \frac{1}{R_{in}} + \frac{N^2}{R_{load}} \right) \frac{dv_i(t)}{dt} + v_i(t) = \frac{L}{R_{in}} v_0 \delta(t) \]

Solving an equation like this is most easily done with

LaPlace Transforms, and we don’t want to do that (yet!) in 6.002. So let’s use intuition to solve this equation.

We know our homogeneous solution will take the form

\[ A e^{-\tau C} \] where \( C = \frac{1}{R_{in}} + \frac{\frac{N^2}{R_{load}}} \]

Since \( v_i \) is driven by a delta function in our equation, we can use that to set initial conditions, and at \( t=0 \), it will be zero. Our particular solution, then, is \( D \) and \( \delta \) will be set by the delta function to \( \frac{v_0 R_{load}}{R_{load} + N^2 R_{in}} \) (Via KCL)

\[ v_i(t) = \frac{v_0 R_{load}}{R_{load} + N^2 R_{in}} e^{-\frac{9\pi^2}{L^2}} \]

and

\[ i_L(t) = \frac{1}{L} \int_{-\infty}^{t} v_i(t) = \frac{1}{L} \int_{0}^{t} A e^{-\frac{\tau^2}{L^2}} = -\frac{1}{L^2} \pi^2 A e^{-\frac{9\pi^2}{L^2}} \]

\[ i_L(t) = \frac{v_0 R_{load} e^{-\frac{9\pi^2}{L^2}}}{R_{load} + N^2 R_{in}} \left( 1 - e^{-\frac{9\pi^2}{L^2}} \right) \]
We can reasonably guess these values by thinking about the time behavior of the circuit—especially the inductor. Physically, unless driven by a current source, we can’t instantaneously change inductor current—so the inductor current will be 0 at time $t = 0$. This forces $V_i$ to start at $V_{in}$. We also know that this voltage sets the rate of current change in the inductor, and that as we pull more current through the inductor, we will drop $V_i$ by $(L - N^2) R_i w$, which will eventually pull the voltage across the inductor to 0.

Combining these expressions with the time constant from the differential equation can give the expression that $V_i$ decays from $V_0$ to 0 with a rate!

See pg. 14 for more...

F) $V_{LOAD}(t) = N V_i(t) = \frac{N V_0 R_{LOAD}}{R_{LOAD} + N^2 R_i w} e^{-\frac{t}{\tau}}$

G) Inductors begin to look like short circuits for time $t \rightarrow \infty$, so the voltage across the inductor will eventually go to 0 with all of $V_{in}$ dropped across $R_i w$. Since $V_{LOAD}$ depends on $V_i$, it falls to 0 with $V_i$. This is why transformers are used with AC power (like what is in our wall outlets!) and not with DC power.
Problem 6.3

4) Begin by reducing the circuits to their Norton/Thévenin Equivalents

\[ I(t) \quad R_C \quad C \quad V(t) \]

\[ I(t) = \frac{R_1}{R_1 + R_2} \cdot I(t) \]

\[ R_N = R_1 + R_2 \]

\[ V_{th} = \frac{V(t)}{R_1 + R_2} \cdot R_2 \]

\[ R_{th} = \frac{R_1}{R_1 + R_2} \]

By KCL:

\[ \frac{V(t)}{R_N} + C \frac{du(t)}{dt} - I_{th}(t) = 0 \]

\[ R_N \frac{du(t)}{dt} + V(t) = I_{th}(t) \cdot R_N \]

By KVL:

\[ V_{th}(t) = R_{th} i(t) + L \frac{di(t)}{dt} \]

\[ \frac{1}{R_{th}} \frac{di(t)}{dt} + il(t) = \frac{V_{th}(t)}{R_{th}} \]

8) Capacitor:

\[ \int_{0}^{t} du + \frac{1}{R_C} \int_{0}^{t} v(t) dt = \frac{1}{C} \int_{0}^{t} i_{th}(t) dt \]

\[ V(t) \bigg|_{t=0}^{t} + 0 = \frac{1}{C} \frac{R_1}{R_1 + R_2} \]

V(t) is finite, so its integral over time goes to 0. It is safe to make this assumption because we know I(t) will deliver a finite amount of charge onto the capacitor, which will create a finite voltage.

Inductor:

\[ \int_{0}^{t} di + \frac{1}{L} \int_{0}^{t} i(t) dt = \frac{1}{L} \int_{0}^{t} v_{th}(t) dt \]

\[ i(t) \bigg|_{t=0}^{t} + 0 = \frac{1}{L} \frac{R_2}{R_1 + R_2} \]

i(t) is finite, so we can treat it the same way as we treated v(t) for the capacitor.
Capacitor

\[ V(0^+)-V(0^-) = \frac{R_1}{C(R_1+R_2)} Q \]

\[ V(0^-) = 0 \text{ from initial conditions} \]

\[ V(0^+) = \frac{R_1Q}{C(R_1+R_2)} \]

As \( t \to \infty \), the capacitor will discharge its voltage over resistors.

\[ V(\infty) = 0 \]

Inductor

\[ i(0^+)-i(0^-) = \frac{R_2}{R_1+R_2} \frac{A}{L} \]

\[ i(0^-) = 0 \]

As \( t \to \infty \), the inductor will discharge its flux through the resistors.

\[ i(\infty) = 0 \]

c) Time constants can be observed by inspection from Thevenin/Norton equivalents.

\[ \tau_{\text{cap}} = R_1C \]

\[ \tau_{\text{ind}} = \frac{L}{R_{th}} \]

d) Constructing from a+b we see

\[ V(t) = \frac{R_1}{R_1+R_2} \frac{A}{L} e^{-t/R_{th}} \]

\[ i(t) = \frac{R_2}{R_1+R_2} \frac{A}{L} e^{-t/R_L} \]

for \( t > 0 \)

and \( V(t) = 0 \) \( i(t) = 0 \) for \( t \leq 0 \).

To make a single expression for all time, multiply the \( t > 0 \) expressions by \( u(t) \).
E) \[ \text{Capacitor} \]

\[
\frac{1}{C} \frac{dQ}{dt} = -\frac{R_1}{R_1 + R_Z} \left( \frac{1}{T_{\text{cap}}} \right) Q \frac{d}{dt} \left( C e^{-\frac{t}{T_{\text{cap}}}} u(t) \right) \quad (1)
\]

\[ + \frac{R_l}{R_1 + R_Z} \frac{Q}{C} e^{\frac{t}{T_{\text{cap}}}} u_0(t) \quad (2) \]

\[ + \frac{1}{R_{\text{nc}}} \left( \frac{R_1}{R_1 + R_Z} \right) \frac{Q}{C} e^{-\frac{t}{T_{\text{cap}}}} u_{-1}(t) \quad (3) \]

By substituting and differentiating,

Knowing that \( T_{\text{cap}} = R_{\text{nc}} \)

\(1 + 3\) will cancel, leaving

\[ \frac{1}{C} \frac{Q}{R_1 + R_Z} u_0(t) = \frac{Q}{C} \frac{R_1}{R_1 + R_Z} e^{-\frac{t}{T_{\text{cap}}}} u_0(t) \]

Since \( u_0 \) only has a value at \( t = 0 \) (call this value \( K \))

\[ \frac{1}{C} \frac{Q}{R_1 + R_Z} (K) = \frac{Q}{C} \frac{R_1}{R_1 + R_Z} (1) (K) \quad \checkmark \]

Similarly for the inductor

\[
\frac{1}{L} \frac{dM}{dt} = -\frac{R_Z}{R_1 + R_Z} \left( \frac{1}{T_{\text{ind}}} \right) \frac{M}{L} e^{-\frac{t}{T_{\text{ind}}}} u_0(t) \quad (1)
\]

\[ + \frac{R_Z}{R_1 + R_Z} \frac{1}{L} e^{\frac{t}{T_{\text{ind}}}} u_0(t) \quad (2) \]

\[ + \frac{R_{\text{ind}}}{L} \left( \frac{R_Z}{R_1 + R_Z} \right) \frac{1}{L} e^{-\frac{t}{T_{\text{ind}}}} u_{-1}(t) \quad (3) \]

\(1 + 3\) will cancel, and since \( u_0 = K \) for \( t = 0 \) and \( 0 \) otherwise

\[ \frac{R_Z}{R_1 + R_Z} \frac{1}{L} \times \frac{R_Z}{R_1 + R_Z} (1) (K) \quad \checkmark \]
F) As we noticed in parts B&D, the rise and fall times go with $C$, which increases with $R_{pu}$. So if we decrease $R_{pu}$, we will see faster devices.

There are two downsides to reducing $R_{pu}$. First, there is increased power drain when the MOSFET is on, which makes the circuit hotter.

More significantly, our output low voltage is

$$V_{OL} < \frac{V_s - R_{OL}}{R_{OL} + R_{pu}}$$

so as we decrease $R_{pu}$, we cannot exceed $V_{OL}$. So we are constrained by:

$$R_{pu} \geq \frac{R_{OL} \left(V_s - V_{OL}\right)}{V_{OL}}$$
Notes on solving 6.002 Differential Equations (from pg. 9)

There are at least five ways to do 6.2 part E that are all acceptable for 6.002. The method provided in the solutions is the closest analogue to the way demonstrated in lecture and in the text, since there is no particular solution to a δ function.

You can, then:

1. Use circuit intuition to avoid solving differential equations. Read the time constants from the differential equation, intuit initial conditions, and skip a lot of the algebra.
   Caut: Dangerous with multiple elements - counterintuitive sometimes!

2. Integrate from \( t = 0^- \) to \( t = 0^+ \), like in 6.3. This method is probably the clearest.

3. Integrate, solve for step response, then differentiate. This method is demonstrated in Prof. Parker's notes.

4. Use Laplace. A δ function transforms to 1, so solving in the s-world is remarkably easy.

5. Use the method described in lecture, find homogeneous solution, then hand-wave as needed for particular solution. Again, like (1), this requires good intuition and is not mathematically rigorous.

As a TA interested in having you all do well in 6.002, I like method 2. I solved it with more intuitive, less mathematical methods to contrast with 6.3's solution. If you feel more comfortable with this style, it does work, and you should at least trust your intuition enough to sanity-check your solution by reasoning about element behavior!