6.002 (Spring 2006)

HW #8

Solutions
(Ex 8.1)

\[ i_{IN} = \frac{1}{m} A (1 + e^{-\frac{V_{IN}(t)}{1000}}) (1 - (-)) \]

\[ V_{IN} \]

\[ V_{IN} = 4V \]

\[ \text{when } t \to -\infty \]

\[ V_{IN}(\infty) = 4V \]

\[ C \text{ open.} \]

\[ t = \infty \]

\[ i_{IN} = \frac{R_1}{R_1 + R_2} \]

\[ V_{IN} = \frac{V_{IN}}{4V} \]

\[ \text{when } t \to +\infty \]

\[ \text{very narrow time period across } t = 0, \]

\[ \text{the capacitor is short for abrupt change.} \]

\[ V_{IN} = \frac{V_{IN}}{4V} \]

\[ \Rightarrow V_{IN} = \frac{V_{IN}}{4V} \]

\[ \Rightarrow R_1 = 2k\Omega \]

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\[ \text{from (1)} \]

\[ i_{IN} = \frac{V_{IN}}{R_1 + R_2} = \frac{1}{m} A \Rightarrow \frac{R_1 + R_2}{R_1 R_2} = 4k\Omega \]

\[ \Rightarrow R_2 = 2k\Omega \]

\[ \Rightarrow R_1 = R_2 = 2k\Omega \]
The time constant is: \( \text{time const} = 1 \mu \text{sec} = R_2 \cdot C = R_{\text{eq}} \cdot C \)

\[ R_{\text{eq}} = R_1 \frac{1}{1 + \frac{R_1}{R_2}} \]

\[ 1 \mu \text{sec} = R_{\text{eq}} \cdot C = \frac{R_1}{R_2} \cdot C = 1 \\text{(for } R_2 \text{)} \]

\[ \Rightarrow C = \frac{1 \mu \text{sec}}{R_2} = 1 \mu \text{F} \]

**Ex 8.2:**

\[ V_{\text{Vin}} = 4 \, \text{V} \]

\[ V_{\text{Vin}}(t) \]

\[ t = \infty \]

\[ V_{\text{Vin}}(\infty) = 4 \, \text{V} \]

\[ I_{\text{2n}}(t) = \frac{4 \, \text{V}}{1 \, \text{K}} = 4 \, \text{mA} \]

**B**

When \( t \to \infty \), \( R \) is open for abrupt change.

\[ I_{\text{2n}}(t) = \frac{4 \, \text{V}}{(3 + 1) \, \text{K}} = 1 \, \text{mA} \]

\[ 4 \, \text{V} \]

\[ 1 \, \text{K} \]

\[ 3 \, \text{K} \]

**C**

Initially, \( I_{\text{2n}}(0) = 0 \) because the circuit was in rest.

We note that the circuit is first order RL circuit therefore we can easily figure out the response using time constant.
\[ \frac{1}{R} = \frac{3 \text{ mH}}{R_{eq}} \]

\[ R_{eq} = \frac{k_2}{1 + \frac{3}{k_2}} \]

\[ R_2 = \frac{3 \text{ mH}}{\frac{3}{k_2}} = 4 \text{ mH} \]

\[ I_{\text{IN}}(t) = \left\{ 4 \text{ mA} + (1 \text{ mA} - 4 \text{ mA}) e^{-t/4 \text{ mH}} \right\} \text{ Unit} \]
Answer:

(A) For this case, the circuit looks like the following:

Examine the circuit, we can see that \( V = \frac{\text{di}}{\text{dt}} \). Integrating this function gives \( i(t) = \frac{V}{L} + c \), where \( c \) is a constant of integration. Since \( i(0) = 0 \), \( c \) must be zero. Since \( v(0) = 0 \) and the capacitor is disconnected from the circuit, there's no way for the capacitor to charge, and \( v(t) \) must be zero during this time period. Therefore, our answers to Part (A) are:

\[
\begin{align*}
    i(t) &= \frac{V}{L} \quad \text{when} \quad 0 \leq t \leq T_1 \\
    v(t) &= 0
\end{align*}
\]

(B) The circuit looks like the following:

The only part we have to worry about is the loop with the inductor and the capacitor. We know that the inductor will be discharging the energy it has stored in part (A) into the capacitor in this circuit configuration. The voltage across both elements is the same, and we can write \( v = \frac{\text{di}}{\text{dt}} \) and \( -i = \frac{\text{dv}}{\text{dt}} \). For practice, let's plug through this solution all the way from the differential equation step-by-step. Combining the information above, we can write

\[
LC\frac{d^2v}{dt^2} + v = 0 \quad \text{or} \quad \frac{d^2v}{dt^2} + \frac{1}{LC}v = 0
\]

A good guess for the solution would be something of the form \( v = Ae^{s(t-T_1)} + Be^{-s(t-T_1)} \). We need the \( t-T_1 \) term because we're starting at time \( T_1 \) in this part of the problem. You only need to substitute one term or the other into the equation to get the right quadratic equation (do the math to convince yourself!), so let's use \( v = Ae^{s(t-T_1)} \) for the substitution. Substituting this “solution” and canceling out terms gives a quadratic equation in terms of \( s \):

\[
s^2 + \frac{1}{LC} = 0
\]
Therefore, we can write that our solution is of the form

\[ v(t) = Ae^{j\sqrt{\frac{1}{LC}}(t-T_1)} + Be^{-j\sqrt{\frac{1}{LC}}(t-T_1)} \]

Hopefully, by now you are experienced enough to know that the above behavior is oscillatory, but let’s actually work through the math to get this expression into sines and cosines.

Let’s rewrite \( v(t) \) in the following form:

\[ v(t) = \left( \frac{A+B}{2} \right) \left( e^{j\sqrt{\frac{1}{LC}}(t-T_1)} + e^{-j\sqrt{\frac{1}{LC}}(t-T_1)} \right) + \left( \frac{A-B}{2} \right) \left( e^{j\sqrt{\frac{1}{LC}}(t-T_1)} - e^{-j\sqrt{\frac{1}{LC}}(t-T_1)} \right) \]

By now, you should see that it’s clear how we can use Euler’s relationships to transform \( v(t) \) into sines and cosines.

\[ v(t) = (A+B) \cos \left( \frac{(t-T_1)}{\sqrt{LC}} \right) + j(A-B) \sin \left( \frac{(t-T_1)}{\sqrt{LC}} \right) \]

Now let’s use our initial conditions to solve for \( A \) and \( B \). We know that \( v(T_1) = 0 \) and \( i(T_1) = \frac{VT_1}{L} \). Plugging our initial condition for the voltage into the equation for the voltage tells us that \( A + B = 0 \). Our other condition tells us something about the current, so we need to somehow transform our voltage expression into a current expression. Remember that \( -i = C \frac{dv}{dt} \). So, let’s take the derivative of the voltage expression and multiply by \( -C \).

\[ i(t) = \sqrt{\frac{C}{L}}(A+B) \sin \left( \frac{(t-T_1)}{\sqrt{LC}} \right) - j \sqrt{\frac{C}{L}}(A-B) \cos \left( \frac{(t-T_1)}{\sqrt{LC}} \right) \]

Plugging in the initial condition gives \( A - B = -j \frac{VT_1}{\sqrt{LC}} \).

Now let’s rewrite \( v(t) \) and \( i(t) \) using what we know about \( A + B \) and \( A - B \).

\[ v(t) = \frac{-VT_1}{\sqrt{LC}} \sin \left( \frac{(t-T_1)}{\sqrt{LC}} \right) \]

\[ i(t) = \frac{VT_1}{L} \cos \left( \frac{(t-T_1)}{\sqrt{LC}} \right) \]

This would be our final answer if we allowed the switch to remain closed for all time. However, the switch goes back to being open as soon as \( i(t) \) reaches zero for the first time. Since \( i(t) \) is a cosine function, the switch opens when the argument of the cosine function is equal to \( \frac{\pi}{2} \), or \( \frac{t-T_1}{\sqrt{LC}} = \frac{\pi}{2} \). Solving this equation for \( t \) gives us the value of \( T_2 \).

\[ T_2 = T_1 + \frac{\pi}{2} \sqrt{LC} \]

Therefore, our answer to (B) is the following:

\[
\begin{align*}
  v(t) &= -\frac{VT_1}{\sqrt{LC}} \sin \left( \frac{(t-T_1)}{\sqrt{LC}} \right) \\
  i(t) &= \frac{VT_1}{L} \cos \left( \frac{(t-T_1)}{\sqrt{LC}} \right)
\end{align*}
\]

when \( T_1 \leq t \leq T_2 = T_1 + \frac{\pi}{2} \sqrt{LC} \)
For Part (C) our circuit looks like the drawing below:

\[ \begin{array}{c}
V \\
A \\
C \\
\hline
i(t) \\
C \\
- \\
v(t) \\
\hline
\end{array} \]

During time \( T_1 \) to \( T_2 \), the current through the inductor drops to zero and the capacitor gets charged. Therefore, the initial conditions for Part (C) are \( i(T_2) = 0 \) and \( v(T_2) = -\frac{VT_1}{\sqrt{LC}} \) (The sine term is equal to 1 when evaluated at \( T_2 \)).

Since there is no current flowing in the circuit for Part (C), there is no way to discharge the charge on the capacitor, and therefore the circuit continues to look as it did at \( t = T_2 \) during the entire interval from \( T_2 \) to \( T_3 \).

So, the answer to (C) is the following:

\[
\begin{align*}
v(t) &= -\frac{VT_1}{\sqrt{LC}} \\
i(t) &= 0
\end{align*}
\]

\( \text{when } T_2 \leq t \leq T_3 \)

(D) Now our circuit looks like the one from Part (A), except with slightly different initial conditions. From Part (C), we know that \( i(T_3) = 0 \) and \( v(T_3) = -\frac{VT_1}{\sqrt{LC}} \). Since the initial condition for \( i(t) \) is the same as the one for Part (A), the solution for \( i(t) \) here is the same for Part (A), compensated for the time shift: \( i(t) = \frac{\sqrt{L}}{\sqrt{LC}(t-T_3)} \). Since there is still no current path for the capacitor to discharge its stored charge, the voltage across the capacitor remains the same for the entire interval \( T_3 \) to \( T_4 \).

Therefore, our answer for Part (D) is the following:

\[
\begin{align*}
v(t) &= -\frac{VT_1}{\sqrt{LC}} \\
i(t) &= \frac{\sqrt{L}}{\sqrt{LC}(t-T_3)}
\end{align*}
\]

\( \text{when } T_3 \leq t \leq T_4 \)

(E) After the switch at \( T_4 \), the circuit looks again like it did in part (B). Since none of the elements have changed, we know that the solution will be of the same form as (B), but with different values for the coefficients. Therefore, we can directly write

\[
\begin{align*}
v(t) &= (A + B) \cos \left( \frac{t - T_4}{\sqrt{LC}} \right) + j(A - B) \sin \left( \frac{t - T_4}{\sqrt{LC}} \right) \\
i(t) &= \sqrt{L} \frac{C}{L} (A + B) \sin \left( \frac{t - T_4}{\sqrt{LC}} \right) - j \sqrt{L} C (A - B) \cos \left( \frac{t - T_4}{\sqrt{LC}} \right)
\end{align*}
\]

However, the \( A \) and \( B \) here are different than those in Part (B) because of the differing initial conditions. Our initial conditions are now \( v(T_4) = -\frac{VT_1}{\sqrt{LC}} \) and \( i(T_4) = \frac{\sqrt{L}}{C(T_4 - T_3)} \).
These initial conditions allow us to solve for \( A + B \) and \( A - B \) in the same way we did in Part (B), giving the expressions for \( v(t) \) and \( i(t) \).

\[
v(t) = -\frac{VT_1}{\sqrt{LC}} \cos \left( \frac{t - T_4}{\sqrt{LC}} \right) \quad \frac{V(T_4 - T_3)}{\sqrt{LC}} \sin \left( \frac{t - T_4}{\sqrt{LC}} \right)
\]

\[
i(t) = -\frac{VT_1}{L} \sin \left( \frac{t - T_4}{\sqrt{LC}} \right) + \frac{V(T_4 - T_3)}{L} \cos \left( \frac{t - T_4}{\sqrt{LC}} \right)
\]

We need to find the time when \( i(t) = 0 \), which is defined to be \( T_3 \). Setting the above equation equal to zero and simplifying gives

\[
\tan \left( \frac{T_3 - T_4}{\sqrt{LC}} \right) = \frac{T_4 - T_3}{T_1}
\]

Solving for \( T_3 \) gives the following:

\[
T_3 = T_4 + \sqrt{LC} \arctan \left( \frac{T_4 - T_3}{T_1} \right)
\]

Our final answer for Part (E) is the following:

\[
\begin{align*}
v(t) &= -\frac{VT_1}{\sqrt{LC}} \cos \left( \frac{t - T_4}{\sqrt{LC}} \right) - \frac{V(T_4 - T_3)}{\sqrt{LC}} \sin \left( \frac{t - T_4}{\sqrt{LC}} \right) \quad \text{when} \quad T_4 \leq t \leq T_5 \\
i(t) &= -\frac{VT_1}{L} \sin \left( \frac{t - T_4}{\sqrt{LC}} \right) + \frac{V(T_4 - T_3)}{L} \cos \left( \frac{t - T_4}{\sqrt{LC}} \right)
\end{align*}
\]

(F) The graphs of \( v(t) \) and \( i(t) \) appear below.

![Graphs of v(t) and i(t)](image-url)
(A) The energy stored in an inductor is given by $E_L = \frac{1}{2} Li^2$. From Problem 8.1 Part (A), we have $i(t_1) = \frac{V}{L}$. Substituting for $i$ in the energy equation and simplifying gives the energy stored in the inductor at $t = T_1$:

$$E_L(t_1) = \frac{V^2T_1^2}{2L}$$

(B) The energy in a capacitor is given by the equation $E_C = \frac{1}{2}CV^2$. Since the energy stored in the inductor at $t = T_1$ is transferred to the capacitor at $t = T_2$, the energy in the capacitor at $t = T_2$ is equal to the energy in the inductor at $t = T_1$:

$$\frac{1}{2}Cv^2 = \frac{V^2T_1^2}{2L}$$

Isolating $v$ gives the voltage across the capacitor at $t = T_2$:

$$v = \pm \frac{VT_1}{\sqrt{LC}}$$

Here we choose the minus sign due to the direction of $i(t)$ at $T_1$.

From Problem 8.1 Part (B), we have

$$v(t) = -\frac{VT_1}{\sqrt{LC}} \sin \left( \sqrt{\frac{t - T_1}{LC}} \right)$$

We also know that $T_2 = T_1 + \frac{\pi}{2}\sqrt{\frac{1}{LC}}$. Substituting this in for $t$ in the equation for $v(t)$ gives

$$v(T_2) = -\frac{VT_1}{\sqrt{LC}}$$

which matches what we found by solving for the energy across the capacitor.
(C) From Problem 8.1 Part (D), we have that \( i(t) = \frac{V(t-T_3)}{L} \) for \( T_3 < t < T_4 \). Therefore, \( i(T_4) = \frac{V(T_4-T_3)}{L} \). The energy stored in an inductor is \( \frac{1}{2}L i^2 \), so the energy at \( t = T_4 \) is the following:

\[
E_{L}(T_4) = \frac{V^2(T_4-T_3)^2}{2L}
\]

(D) Conservation of energy requires that the energy stored in the capacitor at \( t = T_5 \) must be equal to the energy stored in the capacitor at \( t = T_2 \) plus the energy stored in the inductor at \( t = T_4 \). Therefore, the energy stored in the capacitor at \( t = T_5 \) is given by the following:

\[
E_{C}(T_5) = \frac{V^2T_1^2}{2L} + \frac{V^2(T_4-T_3)^2}{2L}
\]

Since the energy stored in the capacitor at \( t = T_5 \) is \( \frac{1}{2}CV^2 \), we can set the above equation equal to that and solve for \( V \).

Solving for \( v(T_5) \) gives:

\[
v(T_5) = \pm \frac{V}{\sqrt{LC}} \sqrt{T_1^2 + (T_4-T_3)^2}
\]

Again we choose the minus sign.

From Problem 8.1 Part (E) the voltage across the capacitor at \( t = T_5 \) is equal to \( -\frac{V T_1}{\sqrt{LC}} \cos(\theta) - \frac{V(T_4-T_3)}{\sqrt{LC}} \sin(\theta) \), where \( \theta = \arctan\left( \frac{T_3-T_1}{T_4-T_3} \right) \). Going back to basic trigonometry, we can write that \( \cos(\theta) = \frac{T_1}{\sqrt{T_1^2 + (T_4-T_3)^2}} \) and \( \sin(\theta) = \frac{(T_3-T_1)}{\sqrt{T_1^2 + (T_4-T_3)^2}} \). Substituting these values for the sine and cosine functions in our expression shows that the voltage across the capacitor is the same as we calculated above:

\[
v(T_5) = -\frac{V}{\sqrt{LC}} \sqrt{T_1^2 + (T_4-T_3)^2}
\]

(E) Since there is no way for the capacitor to dissipate its energy during the switching cycle, its charge and voltage will keep building up in the same way it did between \( t = 0 \) and \( t = T_2 \) in the work we did above. Therefore, \( v \) at the end of the \( n \)th switching cycle is given by the following:

\[
v = -\frac{VT\sqrt{n}}{\sqrt{LC}}
\]
Pr. 8.3

From inductor current
\[ V_L = L \frac{di_L}{dt} \]
\[ \int^t_{-\infty} V_L(t) \, dt = \frac{1}{2} \int^t_{-\infty} (V_{mR}(t) - e_{mL}(t)) \, dt \]

From resistor current
\[ i(t) = j R(t) = \frac{E(t) - V(t)}{R} \]
\[ \frac{1}{L} \int^t_{-\infty} (V_{mR}(t) - e_{mL}(t)) \, dt = \frac{E(t) - V(t)}{R} \]

By taking derivative both side,
\[ \frac{1}{L} (V_{mR}(t) - e_{mL}(t)) = \frac{d}{dt} \frac{E(t)}{R} - \frac{1}{R} \frac{d}{dt} \frac{V(t)}{R} = 0 \]

From capacitor current
\[ i(t) = \frac{1}{C} \frac{dV_c(t)}{dt} \]
\[ \frac{V(t) - V_c(t)}{R} = \frac{dV_c(t)}{dt} \]

From (1) and (2),
\[ e(t) = R \left( \frac{1}{C} \frac{dV_c(t)}{dt} + V_c(t) \right) \]
\[ \frac{1}{L} (V_{mR}(t) - e(t)) = \frac{1}{R} \frac{d}{dt} \frac{E(t)}{R} - \frac{1}{R} \frac{d}{dt} \frac{V(t)}{R} \]
\[ \begin{align*}
\dot{V}_{c}(t) &= -R \left( \frac{d}{dt} V_{c}(t) - V_{c}(t) \right) \\
&= \frac{1}{R} \frac{d}{dt} \left( R \left( \frac{V_{c}(t)}{L} \right) + V_{c}(t) \right) - \frac{1}{R} \frac{d}{dt} V_{c}(t) \\
&= C \frac{d^{2}}{dt^{2}} V_{c}(t) \\
\frac{1}{L} V_{m}(t) - \frac{R}{L} \frac{d}{dt} V_{c}(t) - \frac{1}{L} V_{c}(t) &= C \frac{d^{2}}{dt^{2}} V_{c}(t) \\
\frac{1}{LC} V_{m}(t) - \frac{R}{L} \frac{d}{dt} V_{c}(t) - \frac{1}{LC} V_{c}(t) &= \frac{d^{2}}{dt^{2}} V_{c}(t) \\
\frac{d^{2}}{dt^{2}} V_{c}(t) + \frac{R}{L} \frac{d}{dt} V_{c}(t) + \frac{1}{LC} V_{c}(t) &= -\frac{1}{LC} V_{m}(t) \\
\end{align*} \]

\[ \begin{align*}
\dot{V}_{c}(t) &= C \frac{d}{dt} V_{c}(t) \Rightarrow V_{c} = \frac{1}{C} \int \dot{V}_{c}(t) dt \\
\text{Substitute } t_{3} \\
\frac{d^{2}}{dt^{2}} \left( \frac{1}{C} \int \dot{V}_{c}(t) dt \right) + \frac{R}{L} \frac{d}{dt} \left( \frac{1}{C} \int \dot{V}_{c}(t) dt \right) + \frac{1}{LC} \left( \frac{1}{C} \int \dot{V}_{c}(t) dt \right) \\
&= \frac{1}{LC} V_{m}(t) \\
\text{by taking derivative both side and multiplying } C \\
\frac{d^{2}}{dt^{2}} \dot{V}_{c}(t) + \frac{R}{L} \ddot{V}_{c}(t) + \frac{1}{LC} \dot{V}_{c}(t) &= \frac{1}{LC} \dot{V}_{m}(t) \\
\end{align*} \]
\( V_m = V_L + V_R + V_c \) (KVL) — 0

\[ i = i_c = \lambda = i_R \quad \text{(single loop current)} \]

\[ i_c = \frac{dV_c}{dt}, \quad V_L = L \frac{di}{dt}, \quad V_R = iR \]

\[ \Rightarrow V_c = \frac{1}{C} \int i \, dt + V_i = \frac{L}{C} \frac{di}{dt}, \quad V_R = iR \]

Substituting these into 0

\[ V_m = L \frac{di}{dt} + iR + \frac{1}{C} \int i \, dt \]

by taking derivative both sides,

\[ \frac{d}{dt} V_m = L \frac{d^2 i}{dt^2} + \frac{d}{dt} iR + \frac{1}{C} i \]

\[ \frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} i = \frac{1}{L} \frac{d}{dt} V_m. \]

We have the same result that found in A.
\( \text{(c)} \) \( \dot{i}_{L}(t) \) and \( V_{c}(t) \) are states.

\[ V_{L}(t) = V_{IN}- V_{R}(t) - V_{c}(t) \quad \text{using KVL} \quad \text{(1)} \]

\[ \dot{j}_{L}(t) = \dot{i}_{L}(t) \quad \text{(2)} \]

The states \( \dot{i}_{L}(t) \) and \( V_{c}(t) \) have

\[ \frac{d}{dt} \dot{i}_{L}(t) = \frac{1}{L} V_{c}(t) = \frac{1}{L} \left( V_{IN} - V_{R}(t) - V_{c}(t) \right) \quad \text{(3)} \]

\[ \frac{d}{dt} V_{c}(t) = \frac{1}{C} \dot{i}_{L}(t) = \frac{1}{C} \dot{i}_{L}(t) \quad \text{(4)} \]

From (3), \( V_{c}(t) = \frac{1}{C} \int \dot{i}_{L}(t) \, dt \quad \text{(5)} \)

We combine (2) and (3) by substituting (5) into (2)

\[ \frac{d}{dt} \dot{i}_{L}(t) = \frac{1}{L} \left( V_{IN} - R \dot{i}_{L}(t) - \frac{1}{C} \int \dot{i}_{L}(t) \, dt \right) \]

by taking derivative both side

\[ \frac{d^{2}}{dt^{2}} \dot{i}_{L}(t) = \frac{1}{L} \left( \frac{d}{dt} V_{IN} \dot{i}_{L}(t) - R \frac{d}{dt} \dot{i}_{L}(t) - \frac{1}{C} \dot{i}_{L}(t) \right) \]

Using \( \dot{i}_{L}(t) = \dot{c}(t) \)

\[ \frac{d^{2}}{dt^{2}} \dot{c}(t) + \frac{R}{L} \frac{d}{dt} \dot{c}(t) + \frac{1}{C} \dot{c}(t) = \frac{1}{L} \frac{d}{dt} V_{IN}(t) \]
\( V_{IN}(t) = V_U(t) \)

For fast transition \( t = 0^+ \):

- Inductor is open and capacitor starts.

Using Kirchhoff's Voltage Law (KVL):

\[
V = V_{IN} - V_L
\]

At \( t = 0^+ \):

- \( V_L(0^+) = 0 \)
- \( V_C(0^+) = 0 \)
- \( V_R(0^+) = 0 \)
- \( R = 0 \)

\[
e(0^+) = V_R(0^+) + V_C(0^+) = 0
\]

\[
V_L(0^+) = V_{IN}(0^-) - 0 = V
\]

\[
L \frac{d\dot{I}(0^+)}{dt} = V_L(0^-)
\]

\[
\frac{dI}{dt} \bigg|_{t=0^+} = L \frac{V_L(t+0^+)}{L} = \frac{V}{L}
\]
In DC steady state, capacitor is open and inductor is short.

\[ V_{in} = V_{cm} - V_{cm} \]

\[ t = 0 \]

\[ V_{cm} = 0 \] because the op is open

\[ \frac{d^2}{dt^2} x(t) + \frac{R}{L} \frac{dx}{dt} x(t) + \frac{1}{LC} x(t) = \frac{1}{L} V \delta(t) \]

\[ t > 0 \]

\[ \frac{d^2}{dt^2} x(t) + \frac{R}{L} \frac{dx}{dt} x(t) + \frac{1}{LC} x(t) = 0 \quad (1) \]

Let \( j(t) = A \text{ est} \) and plugging into \( (1) \)

\[ s^2 A \text{ est} + \frac{R}{L} s A \text{ est} + \frac{1}{LC} A \text{ est} = 0 \]

\[ s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \]

\[ s = \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \]

Assume \( \frac{R}{2L} < \frac{1}{LC} \)

\[ \alpha = \frac{R}{2L}, \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega = \sqrt{\omega_0^2 - \alpha^2} = \frac{1}{\sqrt{LC}} \frac{R^2}{4L^2} \]

\[ j(t) = A_1 e^{-\frac{R}{2L}t} + A_2 e^{-\frac{R}{2L}t - 3} \frac{1}{\sqrt{LC}} \frac{R^2}{4L^2} \]

or we can rewrite \( j(t) \) as

\[ j(t) = I \sin \left( \omega t - \frac{R}{2L} t + \frac{\pi}{2} \right) e^{-\frac{R}{2L} t} \]

(15)
We know the initial conditions $x(0^+)$, $\frac{dx}{dt} \bigg|_{t=0^+}$.

\[ x(t) = I \sin \left( \frac{1}{LC} - \frac{R^2}{4L^2} \right) t + \phi \right) e^{-\frac{R}{2L} t} \]

\[ x(0^+) = I \sin(\phi) = 0 \Rightarrow \phi = 0 \]

\[ \frac{d}{dt} \left( \phi \right) = I \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \cos \left( \frac{1}{LC} - \frac{R^2}{4L^2} \right) t + \phi \right) e^{-\frac{R}{2L} t} \]

\[ \frac{I}{2L} \sin \left( \frac{1}{LC} - \frac{R^2}{4L^2} \right) t + \phi \right) e^{-\frac{R}{2L} t} \]

Evaluate at $t = 0^+$.

\[ \frac{d}{dt} \left( \phi \right) \bigg|_{t=0^+} = I \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \cos \phi \Rightarrow \frac{V}{L} \]

\[ \cos \phi = 1 \quad (\therefore \phi = 0) \]

\[ I \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{V}{L} \]

\[ I = \frac{V}{L} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \]

\[ \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \]

\[ I = 0, \quad \phi = \frac{R}{2L} \]

\[ \boxed{\frac{V}{L}} \]
Alternatively, we can solve $V_c(t)$ first and compute $i(t)$ from $V_c(t)$, and use different initial conditions $V_c(0^+)$ and $V_A(0^+)$. From (47)

$$\frac{d^2}{dt^2} V_c(t) + \frac{R}{L} \frac{d}{dt} V_c(t) + \frac{1}{LC} V_c(t) = \frac{V_{IN}(t)}{LC}$$

$$+ 0.$$  

$$\frac{d^2}{dt^2} V_c(t) + \frac{R}{L} \frac{d}{dt} V_c(t) + \frac{1}{LC} V_c(t) = \frac{V}{LC}$$

Particular solution \( V_{c_p}(t) = \sqrt{V} \)

Homogeneous solution \( V_{c_h}(t) = Ae^{st} \)

$$\Rightarrow \quad A s^2 e^{st} + \frac{R}{L} Ae^{st} + \frac{1}{LC} Ae^{st} = 0$$

$$s^2 + \frac{R}{L} s + \frac{1}{LC} = 0$$

$$s = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Assume \( \frac{1}{LC} > \frac{R^2}{4L^2} \)

$$s = -\frac{R}{2L} \pm \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

(11)
\[ V_{ch}(t) = V_c \cos (\omega t + \phi) e^{-\alpha t} \]
\[ \omega = \frac{1}{\sqrt{LC}} - \frac{R^2}{4L^2} \]
\[ \alpha = \frac{R}{2L} \]

\[ V_c(t) = V_{ch}(t) + V_{cp}(t) \]

\[ = V_c \cos (\omega t + \phi) e^{-\alpha t} + V \]

Using initial conditions,

\[ V_c(0^+) = V_c \cos (\phi) + V = 0 \quad \text{(A)} \]

\[ j_c(0^+) = \left. \frac{dV_c(t)}{dt} \right|_{t=0^+} = 0 \]

\[ C \left[ -V_c \omega \sin(\omega t + \phi) e^{-\alpha t} \right. \]
\[ \left. -\alpha V_c \cos (\omega t + \phi) e^{-\alpha t} \right] \]
\[ \left| _{t=0^+} = 0 \right. \]

\[ -V_c \omega \sin \phi - \alpha V_c \cos (\phi) = 0 \]

\[ \tan \phi = -\frac{\alpha}{\omega} = -\frac{R}{2L} / (\omega) = -\frac{R}{2L} \cdot \frac{1}{\sqrt{LC} - \frac{R^2}{4L^2}} \]

\[ \phi = -\tan^{-1} \left( \frac{R}{2L} \sqrt{LC} - \frac{R^2}{4L^2} \right) \]
\[ V_c(t) = V_c \cos(\omega t + \phi) \, e^{-\alpha t} + V \]

\[ \phi = -\tan^{-1}\left( \frac{R}{2\epsilon L} \frac{1}{L \epsilon - \frac{R^2}{4\epsilon^2}} \right) \]

\[ V_c(0^+) = V_c \cos(\phi) + V = 0 \]

\[ \cos \phi = \cos \left( -\tan^{-1}\left( \frac{R}{2\epsilon L} \frac{1}{L \epsilon - \frac{R^2}{4\epsilon^2}} \right) \right) \]

\[ \sqrt{L \epsilon \frac{1}{\epsilon} - \frac{R^2}{4\epsilon^2}} \]

\[ V_c = -V \frac{\cos \phi}{\cos \phi} = -\frac{V}{\sqrt{L \epsilon \frac{1}{\epsilon} - \frac{R^2}{4\epsilon^2}}} \]

\[ V_c(t) = V_c \cos(\omega t + \phi) + V \]

\[ \dot{V}_c(t) = \left[ -V_c \cos(\omega t + \phi) e^{-\alpha t} \right. \]

\[ -\alpha V_c \cos(\omega t + \phi) e^{-\alpha t} \left. \right] e^{-\alpha t} \]

\[ = -V_c \left( \cos(\omega t + \phi) + \alpha \cos(\omega t + \phi) \right) e^{-\alpha t} \]

\[ \text{(19)} \]
\( j(t) = -V_0 \cdot C \cdot A \cdot \sin{(\omega t + \phi + \Theta)} \cdot e^{-\alpha t} \)

Where
\( A = \sqrt{\omega^2 + \alpha^2} \)
\( \Theta = \tan^{-1}{\left( \frac{\alpha}{\omega} \right)} \)

\( j(t) = -V_0 \cdot C \cdot A \cdot \sin{(\omega t + \phi + \Theta)} \cdot e^{-\alpha t} \)
\( \phi = \tan^{-1}{\left( \frac{\alpha}{\omega} \right)} \), \( \Theta = \tan^{-1}{\left( \frac{\alpha}{\omega} \right)} \)

\[ \therefore \phi + \Theta = 0 \]

\( A = \sqrt{\omega^2 + \alpha^2} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2} + \left( \frac{R}{2L} \right)^2} = \frac{1}{\sqrt{LC}} \)

\( i(t) = -2V_0 \cdot \frac{1}{RC} \cdot \sin{(\omega t)} \cdot e^{-\alpha t} \)

\[ = \frac{V_0}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \cdot \frac{1}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \cdot \frac{1}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \cdot \sin{\left( \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \right)} \cdot e^{-\alpha t} \]

Note that the answer is the same answer we have previously.
\((F)(t), \mathcal{L}(V_{in}(t)) = 2 \mathcal{L}(V(t))\)

Then \(V_{in}(t) = 2 \mathcal{L}(V(t)) = \frac{d}{dt} \{\mathcal{L}(V(t))\}^3 - 2 = \frac{d}{dt} \{\sqrt{V(t)}\}^3\)

The operation \(2 \frac{d}{dt}(\cdot)\) is a linear operation, therefore, we can use the linear property.

The response to \(\sqrt{V(t)}\) is already proven as

\[ u(t) = I \sin(\omega t) e^{-\alpha t}\]

Taking \(2 \frac{d}{dt}(\cdot)\) operation both side,

\[ i(t) = \frac{d}{dt} \{I \sin(\omega t) e^{-\alpha t}\} = 2 \left\{ I \omega \cos(\omega t) e^{-\alpha t} - \alpha I e^{-\alpha t} \right\} + I \sin(\omega t) (-\alpha e^{-\alpha t}) \]

\[ = 2 I \omega \cos(\omega t) e^{-\alpha t} - \alpha I \sin(\omega t) e^{-\alpha t}\]
Substituting the coefficients,

\[ \alpha = \frac{R}{2L} , \quad \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \]

\[ I = \frac{V}{L \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \]

\[ i(t) = \frac{2V}{L \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \int \frac{R^2}{1 \frac{1}{LC} - \frac{R^2}{4L^2}} \cos\left(\omega t\right) e^{-\frac{R}{2L} t} \]

\[ + \frac{2R}{2L} \cdot \frac{V}{L \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \sin\left(\omega t\right) e^{-\frac{R}{2L} t} \]

\[ = -\frac{2RV}{2L^2} \int \frac{1}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \sin\left(\omega t\right) e^{-\frac{R}{2L} t} \]

\[ + \frac{2V}{L} \cos\left(\omega t\right) e^{-\frac{R}{2L} t} \]