Problem 1

(1) From Table 10.2, \( \delta[n-m] \overset{z}{\leftrightarrow} z^{-m} \)
with a ROC of all \( z \) except 0 (if \( m > 0 \)) or \( \infty \) (if \( m < 0 \)). Therefore, since the \( z \)-transform is linear,

\[
X(z) = 2z^3 - z^{-2} = \frac{2z^5 - 1}{z^2}
\]
with a ROC of all \( z \) except 0 and \( \infty \). Since the ROC includes the unit circle, the Fourier transform exists.

The system has a double pole at \( z = 0 \) and five zeros on the circle of radius \( 2^{-\frac{1}{5}} \). The zeros are all spaced evenly along the circle, and one is located on the real axis at \( z = 2^{-\frac{1}{5}} \). The pole-zero diagram is shown in Figure 1.

Figure 1: Pole-zero plot for sequence 1.

(2) We can rewrite the sequence as

\[
x[n] = 2^n u[n-1] + 4^n u[-n]
= 2 \cdot 2^{n-1} u[n-1] + 4^n u[-n-1] + \delta[n]
\]
From Table 10.2 and the time-shifting property,

\[ 2^n u[n] \leftrightarrow \frac{1}{1 - 2z^{-1}}, \text{ with ROC } |z| > 2 \]

\[ 2^{n-1} u[n - 1] \leftrightarrow \frac{z^{-1}}{1 - 2z^{-1}}, \text{ with ROC } |z| > 2 \]

Also from Table 10.2,

\[ -4^n u[-n - 1] \leftrightarrow \frac{1}{1 - 4z^{-1}}, \text{ with ROC } |z| < 4 \]

\[ \delta[n] \leftrightarrow 1, \text{ with ROC of all } z \]

Therefore, since the \( z \)-transform is linear,

\[
X(z) = \frac{2z^{-1}}{1 - 2z^{-1}} - \frac{1}{1 - 4z^{-1}} + 1 \\
= \frac{2}{z - 2} - \frac{z}{z - 4} + 1 \\
= \frac{2}{z - 2} - \frac{4}{z - 4} \\
= \frac{-2z}{(z - 2)(z - 4)}
\]

The ROC for \( X(z) \) is the intersection of the ROCs for the separate parts calculated above, so the ROC for \( X(z) \) is \( 2 < |z| < 4 \). The ROC does not include the unit circle, so the Fourier transform does not exist.

The system has poles at \( z = 2 \) and \( z = 4 \) and a zero at \( z = 0 \). Figure 2 shows a pole-zero diagram for the system, where the ROC is the shaded region.

![Pole-zero plot for sequence 2.](image)

**Problem 2**

1. From Table 10.2,

\[ \delta[n - m] \leftrightarrow z^{-m} \]

Applying this to each term of \( X(z) \), we find that

\[ x[n] = 12\delta[n - 4] - \delta[n - 1] + 6\delta[n] + 9\delta[n + 2] - 8\delta[n + 5] \]
We start by factoring and expanding $X(z)$ in partial fractions:

$$X(z) = \frac{5}{1 + \frac{1}{3}z^{-1} - \frac{1}{2}z^{-2}}, \text{ with ROC } \frac{1}{3} < |z| < \frac{1}{2}$$

$$= \frac{5}{(1 + \frac{1}{3}z^{-1})(1 + \frac{1}{2}z^{-1})}, \text{ with ROC } \frac{1}{3} < |z| < \frac{1}{2}$$

$$= \frac{2}{1 - \frac{1}{3}z^{-1}} + \frac{3}{1 + \frac{1}{2}z^{-1}}, \text{ with ROC } \frac{1}{3} < |z| < \frac{1}{2}$$

We can then find the inverse $z$-transform by using Table 10.2:

$$\left(\frac{1}{3}\right)^n u[n] \leftrightarrow \frac{1}{1 - \frac{1}{3}z^{-1}}, \text{ for ROC } |z| > \frac{1}{3}$$

$$-\left(-\frac{1}{2}\right)^n u[-n - 1] \leftrightarrow \frac{1}{1 + \frac{1}{2}z^{-1}}, \text{ for ROC } |z| < \frac{1}{2}$$

$$x[n] = 2\left(\frac{1}{3}\right)^n u[n] - 3\left(-\frac{1}{2}\right)^n u[-n - 1]$$

**Problem 3** O&W 10.29(d)

The pole-zero plot indicates that the system has no zeros and two poles at $z = \pm a$, where $a$ is real and $|a| < 1$. Up to a constant multiplicative factor $M$, the $z$-transform has the form

$$H(z) = \frac{M}{(z + a)(z - a)}$$

Assuming the ROC contains the unit circle, the associated Fourier transform is the $z$-transform evaluated at $z = e^{j\omega}$. Dropping the constant multiplicative factor,

$$H(e^{j\omega}) = \frac{1}{(e^{j\omega} + a)(e^{j\omega} - a)}$$

$$= \frac{1}{e^{j2\omega} - a^2}$$

The magnitude of the Fourier transform is then

$$|H(e^{j\omega})| = \frac{1}{\sqrt{(e^{j2\omega} - a^2)(e^{-j2\omega} - a^2)}}$$

$$= \frac{1}{\sqrt{1 + a^4 - 2a^2(\cos(2\omega))}}$$

As $\omega$ increases from 0 to $\frac{\pi}{2}$, the denominator of the expression above increases, so the magnitude of the Fourier transform decreases. As $\omega$ increases from $\frac{\pi}{2}$ to $\pi$, the denominator of the expression above decreases, so the magnitude of the Fourier transform increases. This is shown in Figure 3.
Problem 4 O&W 10.31

From condition 1, since $x[n]$ is real, we know that any complex zeros and poles will come in conjugate pairs. Combined with conditions 2 and 4, we know that the only poles are at $z = \frac{1}{2}e^{j\pi/3}$ and $z = \frac{1}{2}e^{-j\pi/3}$.

Furthermore, since $x[n]$ is right-sided, the ROC is the part of the $z$-plane on the exterior of the circle on which the outermost pole lies. Both poles lie on the $|z| = \frac{1}{2}$ circle, so the ROC is $|z| > \frac{1}{2}$.

Since $X(z)$ has two zeros at the origin, we know that $X(z)$ has the form

$$X(z) = \frac{Mz^2}{(z - \frac{1}{2}e^{j\pi/3})(z - \frac{1}{2}e^{-j\pi/3})}$$

Plugging in $z = 1$, we find that

$$\frac{8}{3} = \frac{M}{(1 - \frac{1}{2}e^{j\pi/3})(1 - \frac{1}{2}e^{-j\pi/3})}$$

$$= \frac{M}{\frac{4}{3} - \cos \left( \frac{\pi}{3} \right)}$$

$$= \frac{4}{3}M$$

$$M = 2$$

Therefore,

$$X(z) = \frac{2z^2}{(z - \frac{1}{2}e^{j\pi/3})(z - \frac{1}{2}e^{-j\pi/3})}, \text{ with } \text{ROC } |z| > \frac{1}{2}$$
(a) To find the system function, we take the $z$-transform of the difference equation:

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} = \frac{z}{z^2 - z - 1} = \frac{z}{(z - \frac{1 + \sqrt{5}}{2})(z - \frac{1 - \sqrt{5}}{2})}$$

The system has a zero at $z = 0$ and two poles at $z = \frac{1 + \sqrt{5}}{2}$ and $z = \frac{1 - \sqrt{5}}{2}$.

Since the system is causal, the ROC is the exterior of the circle on which the outermost pole lies, including infinity. Therefore, the ROC is $|z| > \frac{1 + \sqrt{5}}{2}$.

Figure 4 shows the pole-zero plot for $H(z)$. The ROC is the region outside of the dotted circle.

(b) We will express $H(z)$ in terms of $z^{-1}$ (since the transform tables list transforms in terms of $z^{-1}$), and then calculate the inverse $z$-transform using partial fraction expansion:

$$H(z) = \frac{z}{(z - \frac{1 + \sqrt{5}}{2})(z - \frac{1 - \sqrt{5}}{2})}, \quad |z| > \frac{1 + \sqrt{5}}{2}$$

$$= \frac{z^{-1}}{(1 - \frac{1 + \sqrt{5}}{2}z^{-1})(1 - \frac{1 - \sqrt{5}}{2}z^{-1})}, \quad |z| > \frac{1 + \sqrt{5}}{2}$$

$$= \frac{1}{1 - \frac{1 + \sqrt{5}}{2}z^{-1}} + \frac{-1}{1 - \frac{1 - \sqrt{5}}{2}z^{-1}}, \quad |z| > \frac{1 + \sqrt{5}}{2}$$
Using Table 10.2 to find the inverse $z$-transform produces the following unit sample response:

$$h[n] = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n u[n] - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2}\right)^n u[n]$$

(c) The original ROC does not include the unit circle, so the system is unstable. The ROC that includes the unit circle and therefore guarantees stability is $1 - \frac{\sqrt{5}}{2} < |z| < \frac{1 + \sqrt{5}}{2}$. Using this new ROC to take the inverse $z$-transform of $H(z)$ produces the following stable unit sample response:

$$h[n] = -\frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n u[-n-1] - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2}\right)^n u[n]$$

Problem 6 O&W 10.47

(a) Since $x[n] = (-2)^n$ is an eigenfunction of the system, the first condition implies that $H(-2) = 0$ and that the ROC of $H(z)$ includes $z = -2$.

From the second condition, we can find the system function $H(z)$ in terms of $a$:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{4} z^{-1}}{1 - \frac{1}{4} z^{-1}}$$

$$= \frac{(1 - \frac{1}{2} z^{-1}) (1 + a - \frac{1}{4} z^{-1})}{(1 - \frac{1}{2} z^{-1})}$$

$$= \frac{\left(z - \frac{1}{2}\right) ((1 + a)z - \frac{1}{4})}{z \left(z - \frac{1}{4}\right)}$$

We know that there must be a zero at $z = -2$, so

$$-(1 + a)2 - \frac{1}{4} = 0$$

$$a = -\frac{9}{8}$$

(b) $x[n] = 1 = 1^n$ is an eigenfunction of the system, so the response is

$$y[n] = H(1) \cdot 1^n$$

$$= \frac{1}{1 - \frac{1}{4}}$$

$$= -\frac{1}{4}$$

Problem 7 O&W 10.50

(a) From the pole-zero plot, the system function has the following form (up to some multiplicative constant):

$$H(z) = \frac{z - \frac{1}{a}}{z - a}$$

The Fourier transform is the $z$-transform evaluated at $z = e^{j\omega}$, so therefore

$$|H(e^{j\omega})|^2 = \frac{e^{j\omega} - \frac{1}{a}}{e^{j\omega} - a} \cdot \frac{e^{-j\omega} - \frac{1}{a}}{e^{-j\omega} - a}$$

$$= \frac{1 + \frac{1}{a^2} - \frac{2}{a} \cos(\omega)}{1 + a^2 - 2a \cos(\omega)}$$

$$= \frac{\left(a^2 - 2a \cos(\omega)\right)}{a^2}$$

$$= \frac{1}{a^2}$$
Since $|H(e^{j\omega})|^2$ is constant, $|H(e^{j\omega})|$ is also constant for all $\omega$.

(b) From the law of cosines,
\[
|v_1|^2 = 1^2 + a^2 - 2a\cos(\omega)
\]
\[
|v_1| = \sqrt{1 + a^2 - 2a\cos(\omega)}
\]

(c) From the law of cosines,
\[
|v_2|^2 = 1^2 + \frac{1}{a^2} - 2\frac{1}{a}\cos(\omega)
\]
\[
= \frac{1}{a^2}(a^2 + 1 - 2a\cos(\omega))
\]
\[
|v_2| = \frac{1}{a}\sqrt{1 + a^2 - 2a\cos(\omega)}
\]
\[
= \frac{1}{a}|v_1|
\]

Therefore, $|v_2|$ is proportional to $|v_1|$, independent of $\omega$.

Problem 8

(a) This frequency response corresponds to a system function given by
\[
H(z) = \frac{2z}{(z - \frac{1}{2})(z - \frac{1}{4})} = \frac{2z}{(1 - \frac{3}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} = \frac{8}{1 - \frac{3}{2}z^{-1} + \frac{1}{4}z^{-2}}
\]

The region of convergence must include the unit circle for the Fourier transform to exist in the first place, so the ROC is $|z| > \frac{1}{2}$. Using this to take the inverse z-transform produces the following impulse response:
\[
h[n] = 8 \left( \frac{1}{2} \right)^n u[n] - 8 \left( \frac{1}{4} \right)^n u[n]
\]

(b) We can rewrite the system function as
\[
H(z) = \frac{2z}{(z - \frac{1}{2})(z - \frac{1}{4})} = \frac{2z}{z^2 - \frac{3}{4}z + \frac{1}{8}} = \frac{Y(z)}{X(z)}
\]

We can then convert this into a difference equation:
\[
2zX(z) = z^2Y(z) - \frac{3}{4}zY(z) + \frac{1}{8}Y(z)
\]
\[
2z^{-1}X(z) = Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z)
\]
\[
2x[n - 1] = y[n] - \frac{3}{4}y[n - 1] + \frac{1}{8}y[n - 2]
\]
\[
y[n] = 2x[n - 1] + \frac{3}{4}y[n - 1] - \frac{1}{8}y[n - 2]
\]

To calculate the output $y[n]$ at any time, we just need the previous two values of the output and the previous value of the input. Therefore, an efficient way of calculating $y[n]$ is by storing $y[n - 1]$,
\( y[n-2], \) and \( x[n-1] \) and using the difference equation. Once \( y[n] \) is calculated, the values of \( y[n-2] \) and \( x[n-1] \) can be discarded and \( y[n] \) and \( x[n] \) can then be stored for calculating the next value of the output. This method ensures that only a few additions and multiplications need to be performed at each time step. Since \( x[n] = 0 \) for \( n < 0 \), we initialize all the stored values to be 0.