Problem Set 5 Solutions

Problem 5.1T

For all values of the real parameter $a \in \mathbb{R}$ find the H-Infinity norm of the transfer matrix

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{s}{s^2 + as + 1} \end{bmatrix}. $$

Since $G(j\omega)$ is a diagonal matrix for all $\omega$,

$$\sigma_{\text{max}}(G(j\omega)) = \max\{1, |H_a(j\omega)|\},$$

where

$$H_a(s) = \frac{s}{s^2 + as + 1}. $$

Since

$$|H_a(j\omega)|^2 = \frac{1}{a^2 + (1 - \omega^2)^2/\omega^2}$$

has maximal value of $1/a^2$ at $\omega = 1$, the largest singular value of $G(j\omega)$ over the $j\omega$-axis is

$$\max_{\omega} \sigma_{\text{max}}(G(j\omega)) = \max\{1, 1/a\}. $$

\(^{1}\)Version of April 5, 2001
In addition, the system is stable only for $a > 0$. Hence, the H-Infinity norm of $G$ is given by

$$
\|G\|_\infty = \begin{cases} 
\max\{1, 1/a\}, & a > 0 \\
\infty, & \text{otherwise}
\end{cases}
$$

**Problem 5.2T**

For all values of parameter $T \geq 0$ find the H2 norm of

$$
G(s) = \frac{e^{-Ts} - 1}{s}.
$$

The impulse response of $G$ is the rectangle-shaped signal

$$
g(t) = \begin{cases} 
1, & 0 < t < T \\
0, & \text{otherwise}
\end{cases}
$$

Hence the square of the H2 norm equals

$$
\int_0^\infty |g(t)|^2 dt = \int_0^T dt = T,
$$

and

$$
\|G\|_{H2} = \sqrt{T}.
$$

**Problem 5.3T**

Give an example of a canonical output feedback design setup with scalar state, control, noise, sensor and cost variables, such that there is a single control singularity at $\omega = 0$ and a single sensor singularity at $\omega = \infty$.

Answer (one of many possible):

$$
\dot{x} = -x + u + w,
$$

$$
z = u - x,
$$

$$
y = x.
$$
Problem 5.4T

Give Q-parameterization for the set of all closed loop transfer functions from $w$ to $z$ which can be obtained by using a stabilizing proper LTI controller $C(s)$ the system on Figure 5.1 with

$$P(s) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{s-1}{(s+1)^2} \end{bmatrix}.$$  

![Figure 5.1: Setup for Problems 5.4T and 5.8T](image)

A canonical setup for this problem is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_2 = -C_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Obviously the system can be stabilized by the feedback gain of 2 from $y_1$ to $u_1$. Hence a stabilizing state feedback gain is

$$K = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

A stabilizing observer gain $L$ can be derived using the same idea:

$$L = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$
Hence one can use

\[
T_1(s) := \left( \begin{array}{c|c} A + B_2K & B_2 \\ \hline C_1 + D_{12}K & D_{12} \end{array} \right) = \left[ \begin{array}{cc} 1/(s + 1) & 0 \\ 0 & (s - 1)/(s + 1)^2 \end{array} \right],
\]

\[
T_2(s) := \left( \begin{array}{c|c} A + LC_2 & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array} \right) = \left[ \begin{array}{cc} (s - 1)/(s + 1) & 0 \\ 0 & 1 \end{array} \right].
\]

Finally, since the controller

\[
C(s) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}
\]

stabilizes the system, the resulting closed loop transfer matrix can serve as

**Problem 5.5T**

Matrix \( K \) is such that \( A + BK \) has a single eigenvalue (of multiplicity 3), where

\[
A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}.
\]

Find all possible values of the eigenvalue.

The pair \((A, B)\) is not controllable. The uncontrollable mode corresponds to the left eigenvector

\[
p = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},
\]

and an eigenvalue \( s = 2 \). Hence, the only possible triple eigenvalue of \( A + BK \) is \( s = 2 \).

**Problem 5.6T**

For all values of \( a \in \mathbb{R} \) find the minimum (or infimum) in the abstract H2 optimization problem

\[
\dot{x} = -x + u, \quad x(0) = 1, \quad x(\infty) = 0,
\]
\[
\int_0^\infty (u^2 + 2axu)dt \to \inf.
\]

Completing the squares according to
\[
u^2 + 2axu + 2P(x(u - x) = (u - Kx)^2
\]
yields the Riccati equation
\[
P^2 + 2(a + 1)P + a^2 = 0,
\]
which has a solution
\[
P = \sqrt{1 + 2a - a - 1}
\]
for \(a \geq -0.5\), and this solution is a stabilizing one for \(a > -0.5\). Hence, by the KYP lemma, the minimum equals \(\sqrt{1 + 2a - a - 1}\) for \(a > -0.5\). For \(a = -0.5\), the sequence of controllers for \(a < -0.5\).

**Problem 5.7TT**

Write down a system of LMIs with respect to the scalar parameters \(d_0, d_1\) and, possibly, some other parameters, which has a solution if and only if
\[
d_0 + d_1\omega^2 + \omega^6 > 0 \quad \forall \omega \in \mathbb{R}.
\]

The inequality is equivalent to existence of \(\epsilon > 0\) such that
\[
\Pi(j\omega) = \frac{(d_0 - 1) + (d_1 - 3)\omega^2 - 3\omega^4}{(1 + \omega^2)^3} + 1 \geq \epsilon > 0
\]
for all \(\omega \in \mathbb{R}\). This \(\Pi\) corresponds to the system/quadratic form pair
\[
\dot{x} = Ax + Bu, \quad \sigma(x, u) = \begin{bmatrix} x \\ u \end{bmatrix}' \Sigma \begin{bmatrix} x \\ u \end{bmatrix},
\]
where
\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} d_0 - 1 & 0 & 0 & 0 \\ 0 & d_1 - 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]
The equivalent system of LMI’s is
\[
\begin{bmatrix}
PA + A'P & PB \\
B'P & 0
\end{bmatrix} + \Sigma > 0,
\]
where \(P = P'\) is a symmetric matrix variable.

**Problem 5.8T**

For the setup of Figure 5.1 with
\[
P(s) = (s - 1)^2/(s + 1)^3
\]
give a good lower bound for the H-Infinity norm of the closed loop Since \(P\) has a zero at \(s = 1\), the value of the closed loop sensitivity transfer function \(S = 1 - T\) at \(s = 1\) is 1. Using the Poisson integral formula on the “modified” \(S\), we get
\[
\log |S(a + jb)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a \log |S(j\omega)| d\omega}{a^2 + (b - \omega)^2},
\]
where \(a > 0\). Applying this with \(a = 1\), \(b = 0\) yields
\[
0 \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \log |S(j\omega)| \frac{d\omega}{1 + \omega^2}.
\]
Let \(M\) be the maximum of \(|S(j\omega)|\) for \(\omega > 10\). Using the symmetry \(|S(j\omega)| = |S(-j\omega)|\) and the original assumption \(|S(j\omega)| \leq 0.1\) for \(\omega < 10\), we get
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \log |S(j\omega)| \frac{d\omega}{1 + \omega^2} \leq \log(M) \frac{2}{\pi} \int_{10}^{\infty} \frac{d\omega}{1 + \omega^2} - \log(10) \frac{2}{\pi} \int_{0}^{10} \frac{d\omega}{1 + \omega^2}
\]
\[
= \frac{2}{\pi} (\log(M)(\pi/2 - \arctan(10)) - \log(10) \arctan(10)).
\]
Hence
\[
M \geq 10^\alpha, \text{ where } \alpha = \frac{\arctan(10)}{\pi/2 - \arctan(10)} \approx 14.8,
\]
and \(\|T\|_\infty \geq 10^\alpha - 1\).