Problem 7.1T

Let $M$ be a complex matrix of rank one, i.e. $M = pq'$ where $p, q$ are complex $n$-by-1 matrices and $'$ denotes Hermitian conjugation. Let $\mu_C(M)$ and $\mu_R(M)$ denote the structured singular values $\mu(M, \Delta)$ of $M$ defined with $\Delta = \Delta_C$ and $\Delta = \Delta_R$ respectively, where $\Delta_C$ is the cone of all diagonal matrices with complex entries, and $\Delta_R$ is the cone of all diagonal matrices with real entries.

(a) Express $\mu_C(M)$ and $\mu_R(M)$ as elementary functions of the components $p_i$ and $q_i$ of $p$ and $q$ (i.e. using the elementary operations of multiplication, addition, maximum of $n$ numbers, division, trigonometric, exponential, etc.).

(b) Express the standard upper bounds $\hat{\mu}_C(M)$ and $\hat{\mu}_R(M)$ of $\mu_C(M)$ and $\mu_R(M)$ as elementary functions of $p_i$ and $q_i$.

(c) Which inequalities and/or inequalities relating the numbers $\mu_C(M)$, $\mu_R(M)$, $\hat{\mu}_C(M)$, and $\hat{\mu}_R(M)$ are satisfied for all rank one matrices $M$?

\footnote{Version of May 3, 2001}
Problem 7.2T

Represent the transfer matrix with uncertain parameters

\[ P(s) = \begin{bmatrix} 1/(s + a) & 1/(s^2 + a^2 + b^2) \\ ab/s & 1/(1 + a^2) \end{bmatrix}, \]

where \( a \) ranges over \([-1, 1]\) and \( b \) ranges over \([0, 1]\), in the form

\[ P(s) = P_{11}(s) + P_{12}(s)\Delta(I - P_{22}(s)\Delta)^{-1}P_{21}(s), \]

where \( P_{ij}(s) \) are fixed transfer matrices to be presented with the solution, and \( \Delta \) is an uncertain constant matrix ranging over the set

\[ \Delta = \left\{ \Delta = \begin{bmatrix} \delta_1 I_p & 0 \\ 0 & \delta_2 I_q \end{bmatrix} : \delta_1, \delta_2 \in [-1, 1] \right\}. \]

Problem 7.3TE

Let \( \Delta = \{\Delta\} \) be the cone of 2-by-2 matrices

\[ \Delta = \left\{ \Delta = \begin{bmatrix} r_1e^{j\phi_1} & 0 \\ 0 & r_2e^{j\phi_2} \end{bmatrix} , r_1, r_2 \geq 0, \phi_1, \phi_2 \in [-\pi/6, \pi/6] \right\}. \]

Derive an upper bound \( \hat{\mu}(M, \Delta) \) of \( \mu(M, \Delta) \) such that \( \hat{\mu}(M, \Delta) \leq \hat{\mu}_C(M) \) for all 2-by-2 matrices \( M \), and \( \hat{\mu}(M, \Delta) < \hat{\mu}_C(M) \) at least for one 2-by-2 matrix \( M \) (present this matrix with your solution). Write a MATLAB script to calculate your upper bound and test it on randomly generated 2-by-2 matrices, comparing the values of \( \hat{\mu}(M, \Delta), \hat{\mu}_C(M), \) and \( \hat{\mu}_R(M) \).

Problem 7.4E

Consider a thin wire of unit length. Assume that the temperatures at each of the ends of the wire can be set arbitrarily as functions of time \( u_0(t), u_1(t) \). Temperature distribution \( v = v(r, t) \) is described by the heat equation

\[ v_t = v_{rr} \]

where \( v_t \) is the time derivative of \( v \) and \( v_{rr} \) is the double spatial derivative. For any \( r \in (0, 1) \) let \( y_r(t) = v(r, t) \) be the wire temperature at coordinate \( r \). The objective of this exercise is to build a finite order model and to design a linear controller with temperature actuators \( u_1(t), u_2(t) \) and temperature sensors at \( r = 0.2 \) and \( r = 0.4 \) to control the wire temperature at \( r = 0.6 \) and \( r = 0.8 \).
(a) Derive the transfer matrix $H_r$ from $u = [u_0; u_1]$ to $y_r$, where $0 < r < 1$ is a parameter. **Hint:** find a solution $v$ of the heat equation, such that

$$v(r, t) = e^{st}p(r), \quad v(0, t) = 0, \quad v(1, t) = e^{st}.$$  

Express $H_r$ in terms of $p(r)$, and find $p(r)$ explicitly.

(b) Find zeros and poles of $H_r(s)$ in the right half plane. Give a “common sense” explanation of the right half plane pole locations.

(c) Represent

$$H(s) = \begin{bmatrix} H_{0.6}(s) \\ H_{0.8}(s) \\ H_{0.2}(s) \\ H_{0.4}(s) \end{bmatrix}$$

in the form

$$H(s) = H_0(s) + \Delta(s)$$

where $H_0$ is a 4th order transfer matrix, $\Delta$ is an LTI system with $\|\Delta\|_\infty$ as small as possible.

(d) Using H-Infinity optimization and D-K iteration techniques for design, and simulation for system features verification, design a controller for the controlled output $[y_{0.6}; y_{0.8}]$ to track asymptotically the reference input $r = [r_{0.6}; r_{0.8}]$, so that the closed loop gain in the “sensor noise to controlled output channel” is below the 40dB level, the overshoot in the $r_{0.6}$ to $y_{0.6}$ and $r_{0.8}$ to $y_{0.8}$ channels does not exceed 20 percent, and the time constant of the $r_{0.6}$ to $y_{0.6}$ and $r_{0.8}$ to $y_{0.8}$ channels is as small as you can make it.

**Hint:** use numerically calculated impulse response of $H(s)$ to simulate the closed loop system to check whether it satisfies the overshoot specs.