Lecture 11: H-Infinity optimization\textsuperscript{1}

H-Infinity optimization is similar to H2 optimization in many aspects. However, the H-Infinity approach is more compatible with robustness specifications, which is one of the reasons of its growing popularity in applications.

11.1 Specifics of H-Infinity Optimization

Unlike in the H2 optimization, the basic H-Infinity algorithms solve a suboptimal controller design problem.

11.1.1 Suboptimality

Formally speaking, H-infinity optimization is stated as the problem of designing a stabilizing controller \( C(s) \), which minimizes the H-infinity norm of the closed-loop transfer matrix from \( w \) to \( z \) for a given open loop plant \( P(s) \) (see Figure 11.1). In reality, however, a suboptimal solution is searched for: GIVEN \( \gamma > 0 \) FIND OUT IF THE OBJECTIVE \( \| T_{zw} \|_\infty < \gamma \) CAN BE ACHIEVED. Partially, this is due to the fact that no efficient and numerically robust algorithm for calculating the optimal H-infinty controller is known (up to date, to the best of my knowledge). On the other hand, for a typical setup, the optimal H-infinity controller is not strictly proper.

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11.1.2 What is done by the software

The general state-space format of $P$ is

\[
\begin{align*}
\dot{x} &= Ax + B_1 w + B_2 u \\
z &= C_1 x + D_{11} w + D_{12} u \\
y &= C_2 x + D_{21} w + D_{22} u
\end{align*}
\]

where the coefficient matrices should satisfy the standard non-singularity assumptions:

- for output feedback stabilizability, the pairs $(A, B_2)$ and $(C_2, A)$ must be respectively stabilizable and detectable

- for non-singularity, $D_{21}$ must be right invertible (full measurement noise), $D_{12}$ must be left invertible (full control penalty), and the matrices

\[
\begin{bmatrix}
    A - sI & B_2 \\
    C_1 & D_{12}
\end{bmatrix}, \quad \begin{bmatrix}
    A - sI & B_1 \\
    C_2 & D_{21}
\end{bmatrix}
\]

must be respectively left and right invertible for any $s \in j\mathbb{R}$.

The $\mu$-Analysis and Synthesis Toolbox provides function “hinfsyn” for calculating a suboptimal output feedback:

\[
\text{function } [k,g,gfin]=\text{hinfsyn}(p,nmeas,ncon,gmin,gmax,tol)
\]

(some additional optional inputs and outputs are possible here). Here “p” is the packed model of $P(s)$, “ncon” and “nmeas” are the numbers of actuators
and sensors, “gmin” and “gmax” are a lower and an upper a-priori bounds of the achievable closed-loop H-infinity norm. The program performs a binary search for the “level of optimality” parameter $\gamma$. At any point, it operates with a “current guess” $\gamma$ of the minimal achievable closed loop H-infinity norm, a lower bound $\gamma_-$ and an upper bound $\gamma_+$ of $\gamma$. The initial values of $\gamma_-$ and $\gamma_+$ are supplied by “gmin” and “gmax”, and, initially, $\gamma = \gamma_+$. At each iteration of the algorithm, it is checked whether it is possible to design a controller with the resulting closed loop H-infinity gain less than $\gamma$. If the answer is positive, new values of $\gamma$ and $\gamma_+$ are assigned, according to

$$
\gamma_{\text{new}} = 0.5(\gamma_+ + \gamma_-), \quad \gamma_{\text{new}}^+ = \gamma_-.
$$

If the answer is negative, new values of $\gamma$ and $\gamma_-$ are assigned, according to

$$
\gamma_{\text{new}} = 0.5(\gamma_+ + \gamma_-), \quad \gamma_{\text{new}}^- = \gamma_+.
$$

This process is continued until the relative difference between $\gamma_+$ and $\gamma_-$ becomes smaller than the tolerance parameter “tol”. Then, actual suboptimal feedback design is performed, with $\gamma_+$ being the target H-infinity performance.

For a given $\gamma$, in order to check that a $\gamma$-suboptimal solution exists, the algorithm calls for forming two auxiliary abstract H2 optimization problems. Roughly speaking, one of these equations corresponds to the limitations in the “full information feedback” part of the stabilization task, and the other corresponds to the “zero sensor input” part of the stabilization task. For feasibility of the suboptimal H-infinity control problem, stabilizing solutions $X$ and $Y$ of the resulting Riccati equations must exist and be non-negative definite. In addition, a coupling condition $X^{1/2}YX^{1/2} < \gamma$ must be satisfied.

When checking solvability of the Riccati equations, “hinfsyn” uses the “Hamiltonian matrices approach”, in which the frequency domain condition of existence of a solution $P = P'$ of

$$
PA + A'P + PRP + Q = 0
$$

such that $A + RP$ is a Hurwitz matrix, is expressed in terms of absence of purely imaginary eigenvalues of the auxiliary Hamiltonian matrix

$$
H = \begin{bmatrix}
A & R \\
-Q & -A'
\end{bmatrix}
$$
11.2 Mathematics of H-Infinity Optimization

Solution of the H-Infinity optimization problem is based on the theory of abstract H@ optimization with indefinite quadratic cost forms $\sigma = \sigma(x, u)$.

11.2.1 An algorithm of H-infinity optimization

In this subsection, an algorithm of H-infinity optimization is presented. To keep the formulae simple, a commonly used simplified setup is considered: noises do not enter the cost, no cross-term penalties between control and state variables are allowed, and sensor noises and plant disturbances are decoupled:

$$\dot{x} = Ax + B_1w_1 + B_2u$$

$z = \begin{bmatrix} C_1x \\ u \end{bmatrix}$

$y = C_2x + w_2$

This setup simplifies significantly the formulae for the optimal controller, but derivation of the main result remains essentially the same.

Formal solution of the simplified setup H-infinity optimization problem comes in the form of two coupled Riccati equations: A stabilizing controller with $\|T_{w\rightarrow z}\| < 1$ exists iff the two Riccati equations:

$$AP + PA' + B_1B_1' - B_2B_2' + PC_1C_1'P = 0$$

$$QA + A'Q + C_1'C_1 - C_2'C_2 + QB_1B_1'Q = 0$$

have positive definite stabilizing solutions $P = P'$, $Q = Q'$, such that $P > Q^{-1}$ (or, equivalently, $Q > P^{-1}$).

Note that matrices

$$X = P^{-1}, \quad Y = Q^{-1}$$

are the stabilizing solutions of the Riccati equations

$$XA + A'X + X(B_1B_1' - B_2B_2')X + C_1'C_1' = 0,$$

$$AY + YA' + Y(C_1'C_1 - C_2'C_2)Y + B_1B_1' = 0,$$
which are exactly the equations mentioned in the previous subsection.

One suboptimal controller can be represented in the observer-based form

\[
\begin{align*}
    u &= -B'_2 P^{-1} \dot{x} \\
    \dot{x} &= A \dot{x} + B_1 \dot{w}_1 + B_2 u + WC'_2 (y - C_2 \dot{x}) \\
    \dot{w}_1 &= B'_1 P^{-1} \dot{x}
\end{align*}
\]

where \( W = (Q - P^{-1})^{-1} \), and \( \dot{w}_1 \) represents the “estimate” of the worst possible plant disturbance. The SUBOPTIMAL H-INFINITY CONTROLLER IS IN NO WAY UNIQUE.

11.2.2 Sufficiency: completion of squares

In the solution to the H-infinity optimization problem, Riccati equations also play the role of equations for “completing the squares”. This is, however, a less obvious procedure, compared to the completion of squares in H2 optimization.

For example, one reason to introduce \( P \) is given by

\[
|w_1|^2 - |u|^2 - |C_1 x|^2 = \frac{d}{dt} (x' P^{-1} x) + |w_1 - B'_1 P^{-1} x| - |u + B'_2 P^{-1} x|^2
\]

This explains why \( w_1 = B'_1 P^{-1} x \) is the “worst case plant disturbance”, and why \( u = -B'_2 P^{-1} x \) is a “good” (suboptimal) state feedback.

Sufficiency of the coupled Riccati equations conditions in H-Infinity optimization follows from such “completion of squares” calculations.

11.2.3 Derivation of necessity

To show necessity of the coupled Riccati equations condition, select a specific set of “bad” noises and disturbances:

- for \( t < 0 \), define \( w_2 = -C_2 x_2 \) so that controller output must stay at zero (\( w_1 \) can be selected arbitrarily, subject to the condition that \( x \) in

\[
\dot{x} = A x + B_1 w_1
\]

is square integrable)

- for \( t > 0 \), define \( w_2 = 0, w_1 = -B'_1 p \), where \( p = p(t) \) is a solution of the differential equation

\[
\dot{p} = -A' p + C'_1 q
\]

and both \( p, q \) are square integrable on \((0, \infty)\)
A simple derivation shows that, for the selections made, we have

\[ \|w\|^2 - \|z\|^2 \leq \int_{-\infty}^{0} (|w_1|^2 + |C_2x|^2 - |C_1x|^2)dt \]

\[ + \int_{0}^{\infty} (|q|^2 + |B_2p|^2 - |B_1p|^2)dt - 2p(0)'x(0). \]

The two integrals in this formulae correspond to two abstract H2 optimization problems. The matrices \(P\) and \(Q\) from the solution of the H-infinity optimization problem come exactly from these two H2 optimizations! Hence the greater side of the inequality can be made as small as

\[ x(0)'Px(0) - 2x(0)'p(0) + p(0)'Qp(0). \]

Since the left side must remain positive definite, this implies positivity of \(P,Q\), as well as the coupling condition.
11.3 Mathematical results related to H-Infinity optimization

In this subsection, an effort is made to explain how one comes out with the solution to the H-Infinity optimization problem. Results which historically preceded the “DJKF” solution presented in the previous subsection will be discussed, among them – the Parrott’s lemma and the Nehari’s lemma.

11.3.1 Parrott’s lemma

The following result, called the Parrott’s Lemma apparently deals with a “toy” version of H-Infinity optimization, in which signals are replaced by finite dimensional vectors.

Lemma 11.1 Let $M_{12}, M_{21}, M_{22}$ be three matrices of sizes $q$-by-$k$, $m$-by-$r$, and $q$-by-$r$ respectively. Then the following conditions are equivalent:

(i) the following inequalities hold:

$$\left\| \begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} \right\| < 1, \quad \left\| \begin{bmatrix} M_{21} \\ M_{22} \end{bmatrix} \right\| < 1,$$

where $\|L\|$ denotes the operator norm (the largest singular value) of $L$;

(ii) there exists an $m$-by-$k$ matrix $K$ such that

$$\left\| \begin{bmatrix} K \\ M_{21} \\ M_{22} \end{bmatrix} \right\| < 1.$$

Moreover, if condition (i) is satisfied, the matrix $K$ in (ii) can be defined by

$$u_* = Kw_1,$$ where

$$u_*(w_1) = \arg \min_u \max_{w_2} (\|u + M_{12}w_2\|^2 + |M_{21}w_1 + M_{22}w_2|^2 - |w_1|^2 - |w_2|^2).$$

The setting in the Parrott’s lemma is strikingly similar to the setting of H-Infinity optimization: the “control variable” $u$ is used to insure that the “sensitivity” (the Euclidean norm gain) of the “outputs” $z_1 = u + M_{12}w_2$ and $z_2 = M_{21}w_1 + M_{22}w_2$ with respect to the “noise” $w = [w_1; w_2]$ is less than 1, subject to the “actuation constraints” ($u$ does not enter the expression for $z_2$) and “sensor constraints” ($u$ is not allowed to depend on $w_2$, and hence must be a function of the “sensor output” $y = w_1$).

It is useful to formulate the following generalization of the “Parrott’s lemma”, which has numerous applications in modern theory of LTI system design.
Theorem 11.1 Let \( \sigma : \mathbb{R}^q \times \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R} \) be a quadratic form \( \sigma = \sigma(p, y, u) \) which is strictly convex in \( u \) (i.e. \( \sigma(0, 0, u) \) is a positive definite quadratic form). Then the following conditions are equivalent

(i) the quadratic forms \( \sigma(p, 0, 0) \) and

\[
\sigma_1(p, y) = \min_u \sigma(p, y, u)
\]

are negative definite;

(ii) there exists matrix \( K \) such that the quadratic form \( \sigma_K(p, y) = \sigma(p, y, Ky) \) is negative definite.

Moreover, if condition (i) is satisfied, the matrix \( K \) in (ii) can be defined by \( u^* = Ky \), where

\[
u_* = u_*(y) = \arg \min_u \max_p \sigma(p, y, u).
\]

To get the Parrott’s Lemma 11.1 from Theorem 11.1, use

\[
\sigma(p, y, u) = |u + M_{12}p|^2 + |M_{21}y + M_{22}p|^2 - |p|^2 - |y|^2.
\]

Note that the implication (ii)\( \Rightarrow \) (i) in Theorem 11.1 is trivial. To prove the inverse implication, use the minimax theorem, applied to the quadratic form \( \sigma(p, y, u) \) as a function of \( p \) and \( u \) (by (i), \( \sigma \) will be strictly concave in \( p \) and strictly convex in \( u \)). This yields

\[
\max_p \min_u \sigma(p, y, u) = \min_u \max_p \sigma(p, y, u) \quad \forall \ y.
\]

By (i), the left side of the equality is negative definite, and hence so is the right side. By in “min-of-max” the optimal \( u \) is a linear function of \( y \): \( u = Ky \). (Here we use the fact that minimum/maximum (with respect to \( a \)) of a quadratic form \( f(a, b) \) (which is strictly convex/concave with respect to \( a \)) is a quadratic function of \( b \), and the argument of the minimum/maximum is a linear function of \( b \).) Hence, \( \max_y \sigma(p, y, Ky) \) is negative definite with respect to \( y \).
11.3.2 Nehari’s problem

In this subsection, we explore the H-Infinity optimization in the “frequency domain”.

According to the Q-parameterization, H-Infinity suboptimal controller design problem can be written in the form

$$\|T_0 + T_1QT_2\|_\infty < 1,$$

where $Q$ is the stable transfer matrix to be chosen. Note that the non-singularity condition means that, for some $\epsilon > 0$, the inequalities

$$T_1(j\omega)'T_1(j\omega) \geq \epsilon I, \quad T_2(j\omega)T_2(j\omega)' \geq \epsilon I \quad \forall \omega$$

are satisfied. Hence, according to the KYP Lemma (though some extra work is needed to show this), there exist square transfer matrices $\Psi_1, \Psi_2$, with no unstable zeros and poles in the extended right half plane, and square transfer matrices $V_1(s), V_2(s)$ (which may be unstable, but satisfy the conditions $V_1'V_1 = I, V_2V_2' = I$ on the imaginary axis), such that

$$T_1 = V_1 \begin{bmatrix} \Psi_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} \Psi_2 & 0 \end{bmatrix}V_2.$$

Therefore, the original suboptimality condition can be re-written in the form

$$\left\| \begin{bmatrix} Q_1 + M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \right\|_\infty < 1,$$

(11.1)

where

$$Q_1 = \Psi_1Q\Psi_2$$

is the modified Q-parameter, and $M_{ij}$ are given transfer matrices (not necessarily stable), depending on $T_0, V_1, V_2$.

The problem of finding a stable $Q$ which minimizes the H-Infinity norm in (11.1) is called the Nehari problem.