Lecture 4: Design By Pole Placement

This lecture is devoted to a simple modern control design tool – pole placement with observer-based output feedback.

We will start with formulating the classical problem of pole placement. To develop a solution to the problem, a study of control authority in state space LTI models will be undertaken. The result will be a change of variables that reduces the input-to-state dynamics of any such model to a canonical form of a simple chain of integrators. A similar “dual” transformation will be derived to represent the sensor-to-state map as a chain of differentiations. The idea of separation will be introduced, suggesting that an output feedback controller can be designed by combining a “full information controller” (i.e. a controller which has immediate access to the information about all states of the system) with a “state observer” (a subsystem which attempts to reconstruct the current states using the information about past measurements and control inputs). The resulting output feedback controller is called “observer-based”, or “model-based”. It turns out that the design of full state controllers and state observers is transparent for systems reduced to canonical forms. The set of closed loop poles of systems with observer-based feedback turns out to be a concatenation of the sets of poles of the observer and full state feedback.

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4.1 Canonical Forms of State Equations

Before solving feedback stabilization and pole placement problems, one has to learn how to represent a pair \((A, B)\) of coefficient matrices of state equation

\[
\dot{x} = Ax + Bu.
\]  

(4.1)

in a “canonical” form, which makes it easier to place the eigenvalues of \(A + BK\) by choosing \(K\). By applying linear transformations to the state and to the input one can simplify significantly the equations and the consequent pole placement task.

4.1.1 Feedback Transformations

Certain transformations of matrices \(A\) and \(B\) do not change some of the “scalable” features of the system, such as controllability. The so-called “feedback transformation” is of particular interest. It corresponds to using the “substitution”

\[
x = Tx_{\text{new}}, \quad u = Ru_{\text{new}} + Fx_{\text{new}}
\]  

(4.2)

where \(T\) and \(R\) are invertible square matrices, which results in a new state space model

\[
\dot{x}_{\text{new}} = A_{\text{new}}x_{\text{new}} + B_{\text{new}}u_{\text{new}},
\]

where

\[
A_{\text{new}} = T^{-1}AT + T^{-1}BF, \quad B_{\text{new}} = T^{-1}BR.
\]

In other terms, the relation between the new and the old variables can be expressed as

\[
\begin{bmatrix}
x \\
u
\end{bmatrix} = S
\begin{bmatrix}
x_{\text{new}} \\
u_{\text{new}}
\end{bmatrix}, \text{ where } S =
\begin{bmatrix}
T & 0 \\
F & R
\end{bmatrix}.
\]

The resulting relation between the coefficient matrices can be written as

\[
\begin{bmatrix}
A_{\text{new}} & B_{\text{new}} \\
? & ?
\end{bmatrix} = S^{-1}
\begin{bmatrix}
A & B \\
? & *
\end{bmatrix}S,
\]

where ? denotes some matrix of appropriate size, and * denotes any matrix of appropriate size.
It follows directly from the definitions that the pair \((A_{\text{new}}, B_{\text{new}})\) is controllable if and only if the pair \((A, B)\) is controllable, and that the matrix \(A + BK\) is similar to matrix \(A_{\text{new}} + B_{\text{new}}K_{\text{new}}\), where
\[
K_{\text{new}} = R^{-1}(-F + KT),
\]
in the sense that
\[
A_{\text{new}} + B_{\text{new}}K_{\text{new}} = T^{-1}(A + BK)T.
\]
In particular, matrices \(A + BK\) and \(A_{\text{new}} + B_{\text{new}}K_{\text{new}}\) have identical characteristic polynomials, and therefore the state feedback pole placement problem for the pair \((A, B)\) is equivalent to the pole placement problem for \((A_{\text{new}}, B_{\text{new}})\).

### 4.1.2 The case of scalar control

Feedback transformations of the state vector can simplify controller design. At least they make the design more transparent. For example, system (4.1) with a scalar control \(u\) which is fully controllable can be transformed into a canonical form with
\[
A_{\text{new}} = \begin{bmatrix}
0 & 1 & 0 & \ldots \\
0 & 0 & 1 & 0 & \ldots \\
\vdots & & & & \\
0 & \ldots & 0 & 1 \\
0 & \ldots & 0
\end{bmatrix}, \quad B_{\text{new}} = \begin{bmatrix} 0 \\
\vdots \\
0 \end{bmatrix}, \tag{4.3}
\]
using a feedback transformation (4.2), where
\[
T = [C; CA; \ldots; CA^{n-1}]^{-1}, \quad R = 1, \quad F = -CA^nT,
\]
and the row vector \(C\) is such that
\[
CA^kB = 0 \quad \text{for} \quad k < n - 1, \quad CA^{n-1}B = 1.
\]
(C exists and is unique because, by the controllability assumption, the vectors \(B, AB, \ldots, A^{n-1}B\) must form a basis in \(\mathbb{R}^n\).) In a more appropriate language, the particular selection of \(T\) means that the \(k\)-th component \(x_k\) of \(x_{\text{new}}\) is defined as \(x_k = CA^{k-1}x\), and hence
\[
\dot{x}_k = \begin{cases}
x_{k+1} & k < n \\
CA^n x + CA^{n-1}Bu & k = n
\end{cases}
\]
Hence, matrices $R$ and $F$ are selected in such a way that

$$\dot{x}_n = CA^n x + CA^{n-1} Bu = u_{new},$$

and the transformed state space matrices have the form (4.3). Note that (4.3) a valuable representation of the system as a chain of integrators. One can also say that (4.3) shows different degree of control authority over the states $x_k$: the state $x_n$ is the easiest to control, while the states $x_k$ with smaller indexes $k$ can only be controlled through the states with larger indexes.

When the matrices $A, B$ are represented in the “canonical” form (4.3), it is easy to check that the state feedback

$$K_{new} = -\begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \end{bmatrix}$$

produces the closed loop matrix $A + BK$ with the characteristic polynomial

$$p(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0.$$ 

Therefore, pole placement problem becomes trivial, as soon as the system is represented in an appropriate canonical form.

4.1.3 The case of vector control

Now let us formulate a general version of this result for a controllable pair of matrices $(A, B)$ (the “multi-input” version of the single-input canonical form (4.3)).

**Theorem 4.1** Any pair $(A, B)$ can be reduced by a feedback transformation (4.2) to a canonical form

$$A_{new} = \begin{bmatrix} A_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & A_k & 0 \\ 0 & \cdots & 0 & A_u \end{bmatrix}, \quad B_{new} = \begin{bmatrix} B_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & B_k & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \quad (4.4)$$

where the pairs $(A_i, B_i)$ with $i > 0$ are all in the canonical form (4.3). (In the case when the columns of $B$ are linearly independent, the ”zero” column block shown in $B_{new}$ will have zero width – i.e. there will be no ”zero” column block. Similarly, when the rows of $B$ are linearly independent, there will be no ”zero” row block.)
Proof  The statement will be proven by describing a recursive algorithm for finding the coordinate transformation (4.2) (mathematically, this means proving "by induction").

Actually, we will be proving that any pair \((A, B)\) can be transformed into the form

\[
A_{\text{new}} = \begin{bmatrix}
A_u & 0 & \ldots & 0 \\
0 & 0 & E_0 & 0 \\
0 & 0 & 0 & E_1 \\
\vdots & & & \ddots \\
0 & \ldots & & 0 & E_{k-1} \\
0 & 0 & 0 & 0 & E_k
\end{bmatrix}, \quad B_{\text{new}} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
E_k
\end{bmatrix}
\tag{4.5}
\]

where the diagonal zero blocks are square, and each \(E_i\) has the form

\[
E_i = [I_{n(i)} \ 0].
\]

It is easy to check that, by re-arranging the order of the elements of the state vector, any model (4.5) can be transformed into form (4.4).

The following observation will be helpful in deriving the feedback transformation: a composition of two substitutions

\[
u = R_1 u_1 + F_1 x_1, \quad x = T_1 x_1
\]

and

\[
u_1 = R_2 u_2 + F_2 x_2, \quad x_1 = T_2 x_2
\]

is equivalent to a single substitution

\[
u = R_1 R_2 u_2 + (R_1 F_2 + F_1 T_2) x_2, \quad x = T_1 T_2 x_2.
\]

Therefore, in order to show that any controllable pair \((A, B)\) can be transformed into form (4.5), it is sufficient to design a sequence of feedback transformations which gradually simplify the problem, until it is reduced to the desired form.

Given \(B\), we can find square non-singular matrices \(T_0, R_0\) such that

\[
B = T_0 \begin{bmatrix}
0 & 0 \\
I_r & 0
\end{bmatrix} R_0^{-1}.
\]
where $r$ is the rank of $B$. Consider the block partition

$$T_0^{-1}AT_0 = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{F}_1 & \bar{F}_2 \end{bmatrix}$$

where the size of $F_2$ is $r$-by-$r$. In the new coordinates

$$x_{\text{new}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u_{\text{new}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where vectors $x_2$ and $u_1$ have length $r$, and

$$x = T_0 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u = R_0 \begin{bmatrix} u_1 - F_1 x_1 - F_2 x_2 \\ u_2 \end{bmatrix},$$

equations (4.3) reduce to

\[
\begin{align*}
\dot{x}_1 &= \bar{A}x_1 + \bar{B}x_2 \\
\dot{x}_2 &= u_1
\end{align*}
\] (4.6) (4.7)

The idea of the next step is that $x_2$ can be considered as a new “control” variable for the first equation in (4.6). If the pair $(\bar{A}, \bar{B})$ can be reduced to the form (4.5) with $A_{\text{new}} = \bar{A}_{\text{new}}, B_{\text{new}} = \bar{B}_{\text{new}}$, using a feedback transformation with $T = \bar{T}, R = \bar{R}, F = \bar{F}$, then the original system can be reduced to the form (4.5) by using the feedback transformation

\[
\begin{align*}
x_1 &= \bar{T}x_{1\text{new}}, \\
x_2 &= \bar{R}x_{2\text{new}} + \bar{F}x_{1\text{new}}, \\
u_1 &= \bar{R}u_{1\text{new}} + \bar{F}(\bar{A}x_{1\text{new}} + \bar{B}x_{2\text{new}})
\end{align*}
\] (4.8) (4.9) (4.10)

When system (4.1) is in the canonical form (4.4), designing a “feedback matrix” $K$ which “places” the eigenvalues of matrix $A + BK$ into desired locations is easy. One way to do this is by recognizing that system (4.4) can be decomposed into $k$ independent subsystems

$$\dot{x}_i = A_i x_i + B_i u_i$$

(plus an uncontrollable subsystem defined by $A_u$), and using a separate feedback $u_i = K_i x_i$ in each of the subsystems. This is sufficient for most situations,
though, formally speaking, does not allow to obtain an arbitrary closed loop characteristic polynomial. An alternative way is to introduce the extra feedback restriction $u_i = x_{i+1,1}$, where $x_{i+1,1}$ denotes the first component of $x_i$, and $i < k$. Then the system with the remaining scalar control variable $u_k$ will have the form of (4.3), with the standard scalar feedback control pole placement design to follow.

4.2 Observer-Based Controllers

In this subsection, the technique for pole placement in output feedback systems is described, based on the solution to the eigenvalue placement problem given in the previous subsection.

4.2.1 The Pole Placement Problem

The Pole Assignment Problem is defined in terms of the block diagram on Figure 4.1: given a finite order LTI model $P$ (the “plant model”), and a set $\Lambda$ of points in the complex plane, design a finite order LTI model $K$ such that the interconnection is well posed and the poles of the feedback system belong to the set $\Lambda$.

![Figure 4.1: Interconnection as an autonomous system](image)

As it is frequently the case, it is convenient to solve the problem in terms of state space models. We will assume that $P$ is given and has the form

$$P := \begin{pmatrix} A_p & B_p \\ C_p & D_p \end{pmatrix},$$

and that a controller

$$K := \begin{pmatrix} A_K & B_K \\ C_K & 0 \end{pmatrix},$$
where $A_K$ has same size as $A$, is to be designed. (Note that in this case the well posedness of the feedback system is guaranteed, since $D_K = 0$.) The resulting closed loop state space equations are given by

$$\dot{x} = A_{cl}x,$$

where $x = \begin{bmatrix} x_p \\ x_k \end{bmatrix}$, $A_{cl} = \begin{bmatrix} A_p & B_p C_k \\ B_k C_p & B_k D_p C_k \end{bmatrix}$,

where $x_p, x_k$ are respectively the states of the plant and controller. The poles of the closed loop system are, therefore, the eigenvalues of $A_{cl}$.

It can be shown that, if a pole placement problem can be solved at all, it can be solved using a strictly proper controller same order as the plant. On the other hand, it is frequently possible to stabilize a system with a controller of a much lower order, but, up to date, there is no technique for finding such lower order controllers efficiently in the general case.

### 4.2.2 State observer design

By a state observer we mean a specific part of certain feedback control laws, which “mimic” the original system equations. A full state observer for system

$$P := \begin{pmatrix} A_p & B_p \\ C_p & D_p \end{pmatrix},$$

with control input $u$ and sensor output $y$, has the form

$$\dot{\hat{x}} = A_p \hat{x} + B_p u + L(C_p \hat{x} + D_p u - y)$$

(4.11)

where $L(C_p \hat{x} + D_p u - y)$ is the “correction term”, defined by the observer gain $L$. The main benefit of using (4.11) is a complete decoupling of the error dynamics from control and state variables:

$$\dot{e} = (A_p + LC_p)e,$$

where $e = \hat{x} - x$. (4.12)

The problem of assigning the observer poles (i.e. the eigenvalues of $A_p + LC_p$) is the familiar eigenvalue assignment problem for the pair $(A_p', C_p')$, since the characteristic polynomials of $A_p - LC_p$ and $A_p' - C_p'L'$ are the same.

If the pair $(C_p, A_p)$ is observable (i.e. the pair $(A_p', C_p')$ is controllable), the observer poles can be assigned arbitrarily. This, however, is not always desirable, since a large $L$ may lead to a substantial amplification of the sensor noises, which will cancel the effect of good observer poles.
4.2.3 Model-based (observer-based) feedback

The fact that the observer error dynamics (4.12) is completely decoupled from the control and state signals suggests that a state observer and a state feedback can be combined without changing closed loop system poles. More precisely, let the characteristic polynomial of \( A_p + B_p K \) be \( \delta_c(s) \), and let the characteristic polynomial of \( A_p + L C_p \) be \( \delta_o(s) \). Then the feedback controller

\[
\dot{x} = A_p \hat{x} + B_p u + L(C_p \hat{x} + D_p u - y),
\]

i.e. with the state space form

\[
\text{controller} := \begin{pmatrix} \frac{A_p + B_p K + L C_p + L D_p K}{K} & -L \\ 0 & 0 \end{pmatrix}
\]

produces a closed loop system with the characteristic polynomial

\[
\delta(s) = \delta_o(s) \delta_c(s).
\]

This becomes obvious when the closed loop state dynamics is re-written in terms of the original state \( x \) and the state estimation error \( e \):

\[
\begin{align*}
\dot{e} &= (A_p + L C_p)e \\
\dot{x} &= B_p K e + (A_p + B_p K)x
\end{align*}
\]

4.2.4 Discussion

As one can see from the theoretical basics of the method, pole placement is a very rough technique. Essentially, it calls for wiping out the original system structure almost completely (reducing it to a canonical form), and then re-imposing some completely artificial structure upon the system to place the poles. The approach makes it difficult to keep the controller gain down. It should also be remembered that the technique does not work well for systems with many states because of potential numerical instability. The problem is, characteristic polynomials of higher order systems are ugly and not very useful. (For example, what would be the characteristic polynomial of pure delay?)

Here are some suggestions for pole placement design:

- Try to minimize the change in the characteristic polynomial. Do not move the poles which are well placed originally.
• Do not allow the state feedback and observer gains to be large. The procedure is unlikely to give good results if the matrices transforming the original system to a canonical form are poorly conditioned.

• It appears to be more meaningful to apply pole placement when the control (or sensor) are scalars. Try to decompose the system first, and select the state feedback and observer poles separately.