Lecture 6: Feedback Limitations: Integral Form

This lecture deals with feedback design limitations expressed in an integral form. In the basis of such results are the Q-parameterization theorem and the Cauchy integral relation for analytical functions.

6.1 Integral Identities for Analytical Functions

This subsection contains some introductory material on functions of complex variables.

6.1.1 Analytical Functions

Let \( \Omega \) be an open subset of the complex plane \( \mathbb{C} \). A function \( f : \Omega \rightarrow \mathbb{C} \) is called analytical on \( \Omega \) if the limit

\[
f'(s) = \lim_{\delta \neq 0, |\delta| \to 0} \frac{f(s + \delta) - f(s)}{\delta}
\]

exists (and is continuous) for all \( s \in \Omega \).

For example, functions \( f(s) = \exp(s) \) and \( f(s) = \sqrt{s} \) (where, for \( \phi \in (-\pi, \pi) \), \( \sqrt{r \exp(j\phi)} = \sqrt{r} \exp(j\phi/2) \)) are analytical in the right half plane

\[ C_+ = \{ s \in \mathbb{C} : \text{Re}(s) > 0 \} \]

---

\(^1\)Version of February 28, 2001
while \( f(s) = \text{Re}(s) \) and \( f(s) = |s| \) are not.

Important class of analytical functions defined on \( \mathbb{C}_+ \) is \( H_\infty \), which includes, in particular, all stable proper transfer functions, and all Laplace transforms of integrable functions defined on \([0, \infty)\).

### 6.1.2 The Cauchy Identity

Assume that \( \Omega \) is such that a continuous one-to-one transformation of \( \Omega \) into the unit disc

\[
D = \{ s \in \mathbb{C} : |s| < 1 \}
\]

exists. Let \( f : \Omega \to \mathbb{C} \) be an analytical function, and let \( \phi : [0, 1] \to \Omega \) be a differentiable function such that \( \phi(0) = \phi(1) \). Then

\[
\int_0^1 f(\phi(\tau)) \dot{\phi}(\tau) d\tau = 0.
\]

The integral in (6.1) can be interpreted as the contour integral

\[
\int_C f(z) dz = 0,
\]

where \( C = \phi([0, 1]) \) is the (closed) contour traced by \( \phi(t) \) as \( t \) ranges from 0 to 1.

Integral relation (6.1) is frequently combined with the “number of encirclements” identity

\[
\int_0^1 \frac{\dot{\phi}(\tau) d\tau}{\phi(\tau) - z_0} = j(\theta(1) - \theta(0)),
\]

where \( \phi(\tau) \) does not take value \( z_0 \) and hence can be represented in the form

\[
\phi(\tau) = z_0 + r(\tau)e^{j\theta(\tau)},
\]

with \( r(\tau) > 0, \theta(\tau) \) being continuous functions. Note that in (6.2) the quantity \( (\theta(1) - \theta(0))/2\pi \) represents the number of encirclements (anticlockwise) that the path of \( \phi(\tau) \) makes around \( z_0 \) as \( \tau \) increases from 0 to 1.

In particular, when the path of \( \phi \) encircles \( z_0 \) exactly once, (and hence \( z_0 \in \Omega \)), combining (6.1) and (6.2) yields the well-known Cauchy identity

\[
f(z_0) = \frac{1}{2\pi j} \int_C \frac{f(z) dz}{z - z_0}.
\]
6.1.3 The Parceval Identity

When working with continuous time systems, the most important integral relation for analytical functions appears to be the Parceval identity.

Let \( f \in L^2(\mathbb{R}) \), i.e. \( f : \mathbb{R} \rightarrow \mathbb{C} \) is a function which is square integrable over \((-\infty, \infty)\). Then the Fourier transform \( F = F(j\omega) \in L^2(j\mathbb{R}) \) of \( f = f(t) \) exists in the sense that the integral

\[
\int_{-\infty}^{\infty} \left| F(j\omega) - \int_{-T}^{T} e^{-j\omega t} f(t) dt \right|^2 d\omega
\]

converges to zero as \( T \rightarrow \infty \). If \( F, G \in L^2(j\mathbb{R}) \) are the Fourier transforms of \( f, g \in L^2(\mathbb{R}) \) then

\[
\int_{-\infty}^{\infty} \bar{f}(t) g(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(j\omega) G(j\omega) d\omega. \tag{6.3}
\]

One can think of the Parceval identity as a special enhanced version of the Cauchy identity.

6.1.4 The class \( H_2 \)

The class \( H_2 \) consists of the functions which are defined both on the right half plane, as analytical functions on \( \mathbb{C}_+ \) which are Laplace transforms of causal signals \( f \in L^2(\mathbb{R}) \) (i.e. such that \( f(t) = 0 \) for \( t < 0 \)) and on the imaginary axis \( j\mathbb{R} \), as the elements of \( L^2(j\mathbb{R}) \) which are Fourier transforms of causal signals \( f \in L^2(\mathbb{R}) \).

It is important to be able to tell whether a particular function from \( L^2(j\mathbb{R}) \) belongs to the class \( H_2 \), without actually knowing what the inverse Fourier transform is. The following criteria is very useful: an analytical function \( F : \mathbb{C}_+ \rightarrow \mathbb{C} \) is a Laplace transform of a causal signal \( f \in L^2(\mathbb{R}) \) if and only if there exists a finite constant \( c \) such that

\[
\int_{-\infty}^{\infty} |F(\sigma + j\omega)|^2 d\omega < c \tag{6.4}
\]

for all \( \sigma > 0 \), in which case for almost all \( \omega \in \mathbb{R} \) the limit

\[
F(j\omega) = \lim_{\sigma \to 0, \sigma > 0} F(\sigma + j\omega)
\]

exists and equals the Fourier transform of \( f \).
6.2 Integral Identities for Stable Transfer Functions

In this subsection, it is shown how the Cauchy formulae and its analogs are applied to system analysis.

6.2.1 Special features of stable transfer functions

The condition of stability, imposed on closed loop transfer functions, is used to explain many of well-known feedback control limitations. One has to understand that, value-wise, stable transfer functions are very “redundant”. (For example, behavior of such function in a neighborhood of a single point defines it in a unique way.) In feedback system design, one usually works with the following two types of statements:

- Complex frequency response of a stable transfer function can be reconstructed in the whole right half plane from the real part of its values on the imaginary axis. Similarly, the phase of a stable and minimum phase transfer function can be reconstructed when the system gain is known over the whole imaginary axis.

- In order for a stable transfer function \( f(s) \) (or its derivative, etc.) to take large values at some points in the left half plane, the values of \( |f(s)| \) on a significant portion of the imaginary axis must be large.

The statements of the first type are used to show that certain restrictions apply to the closed loop transfer functions no matter what the plant equations are. The statements of the second type are used in conjunction with the Q-parameterization statement. For example, for SISO feedback systems, the Q-parameterization is equivalent to stating that the closed loop sensitivity function must equal 0 at the unstable poles of the plant, and 1 at the unstable zeros of the plant. The, is an unstable pole \( p \) is located very close to an unstable zero \( z \), we have \( S(p) = 0, S(z) = 1 \), and hence the derivative \( dS/ds \) must be very large somewhere near \( p, z \). Hence, \( |S(j\omega)| \) must be large on a substantial portion of the imaginary axis.

The Poisson Integral

For functions from the class \( H_2 \), there is a nice direct integral formula which uses the real part of its values on the imaginary axis to reconstruct the actual values in the right half plane.
**Theorem 6.1** If $F \in H_2$ then

$$F(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Re} F(j\omega)d\omega}{s-j\omega},$$

(6.5)

for all $s \in \mathbb{C}_+$. 

**Proof** Let $F$ be the Fourier transform of $f$. Note that $f$ is causal and define $g(t) = f(t) + f(-t)$. Then, by the definition of $F(s)$,

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt = \int_{-\infty}^{\infty} g(t)\bar{r}(t)dt,$$

where

$$r(t) = u(t)e^{-st},$$

and $u(\cdot)$ denotes the unit step function. Hence, by the Parceval formula,

$$F(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega)R(j\omega)d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Re} F(j\omega)d\omega}{s-j\omega},$$

since the Fourier transform $R(j\omega)$ of $r(t)$ is $R(j\omega) = 1/(j\omega + \bar{s})$ and the Fourier transform $G(j\omega)$ of $g$ is $G(j\omega) = 2\text{Re} F(j\omega)$. 

When looking separately for the real and imaginary part of $F(s)$, (6.5) yields

$$\text{Re} F(a+bj) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\text{Re} F(j\omega)d\omega}{a^2 + (\omega - b)^2};$$

(6.6)

$$\text{Im} F(a+bj) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\omega - b)\text{Re} F(j\omega)d\omega}{a^2 + (\omega - b)^2}. $$

(6.7)

Important versions of the Poisson formula are obtained when $F$ is defined as a logarithm $F(s) = \log H(s)$ of a minimum-phase stable transfer function.
6.2.2 Bode Gain and Phase Relation

We say that a stable rational transfer function $H(s)$ is \textit{minimum phase} if $H(s) \neq 0$ for $\text{Re}(s) > 0$, and $H(1) > 0$. For non-rational transfer functions the definition is not so straightforward. In general a function $H(s)$ from class $H_{\infty}$ is called “minimum phase” if $H(1) > 0$ and any function $F(s)$ from class $H_{2}$ can be approximated arbitrarily well in the mean square sense by the products $H(s)Q_k(s)$, where $Q_k(s)$ are stable rational transfer functions. It can be shown that $H(s) \neq 0$ in the right half plane for any minimum phase function $H(s) \in H_{\infty}$. However, there are functions from $H_{\infty}$ which have no unstable zeros, but are still not minimum phase, such as $H(s) = \exp(-s)$. On the other hand, a function can have a zero on the imaginary axis, but still be minimum-phase such as $H(s) = s/(s+1)$.

The following statement is known as the \textit{Bode’s Gain/Phase relation}.

\textbf{Theorem 6.2} If $L \in H_{\infty}$ is a minimum-phase transfer function, then the phase of $L$ is uniquely defined by its gain, according to the formula:

$$\text{phase}(L(j\omega)) = \int_{-\infty}^{\infty} \frac{d \log |L(e^{j\nu})|}{d\nu} \psi(\nu) d\nu,$$

where

$$\psi(\nu) = \frac{1}{\pi} \log \frac{e^{|\nu|/2} + e^{-|\nu|/2}}{e^{|\nu|/2} - e^{-|\nu|/2}}.$$

Since $\psi(\nu) \geq 0$ and $\psi(\nu) \ll 1$ for $\nu \gg 1$, the main contribution to the integral is made in the region $\nu \approx 0$. Hence the integral mainly depends on the values of

$$\frac{d \log |L(e^{j\nu})|}{d\nu}$$

with $\nu \approx 0$, i.e. essentially on $(d/d\omega) \log L(j\omega)$.

\textbf{Bode’s Sensitivity Integral}

The following relation is an example of an inequality that does not allow the closed loop sensitivity to be small on the imaginary axis.

\textbf{Theorem (Bode’s Sensitivity Integral)} \textit{Let $S$ be a stable rational transfer function such that $1 - S$ has relative degree greater than...}
1. Then

$$\int_0^{\infty} \log |S(j\omega)|d\omega = \pi \sum_{k=1}^{m} \text{Re}(z_k)$$

where $z_k$ are the unstable zeros of $S$.

Here the relative degree condition ensures that $S(s) \approx 1$ for large $|s|$. On the other hand, $S(z_k) = 0$. This means large variation of $S$ in the right half plane, and an inequality bounding $\log |S(j\omega)|$ from below in an “integral” sense results.