Quiz 2 Sample Problems

These are sample questions for quiz 2. The quiz will be open book/notes. The quiz will start at 1.05pm on Wednesday, May 16. You will have at least 85 minutes to solve the problems. If the room will be available after the scheduled end of the class, you will be allowed to continue for at least another hour.

The number of problems in the real quiz 2 will be less than the number of problems in this sample.

Problem Q2S.1

For the canonical LTI feedback setup

\[ \dot{x} = ax + u + w_1, \quad z = \begin{bmatrix} x \\ u \end{bmatrix}, \quad y = x + w_2, \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \]

where \( a \in \mathbb{R} \) is a parameter, find the controller which minimizes the H2 norm of the closed loop transfer matrix from \( w \) to \( z \).

This is a canonical LTI design setup with

\[
A = a, \quad B_1 = [1 \ 0], \quad B_2 = 1, \quad C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_2 = 1, \quad D_{11} = 0, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = [0 \ 1], \quad D_{22} = 0.
\]

The optimal H2 controller is given by

\[
u = K_1 \hat{x}, \quad \dot{\hat{x}} = A \hat{x} + B_2 u + K'_2(C_2 \hat{x} - y),
\]
where $K_1$ is the optimal feedback gain $u = K_1 x$ for the full information control abstract H2 optimization problem

$$\dot{x} = Ax + B_2 u, \quad x(0) = x_0, \quad x(\infty) = 0, \quad \int_0^\infty |C_1 x + D_{12} u|^2 dt \to \min,$$

and $K_2$ is the optimal feedback gain $q = K_2 p$ for the state estimation abstract H2 optimization problem

$$\dot{p} = A' p + C_2' q, \quad p(0) = p_0, \quad p(\infty) = 0, \quad \int_0^\infty |B_1' p + D_{21} q|^2 dt \to \min,$$

subject to the non-singularity assumption. The initial conditions $x_0, p_0$ do not affect $K_1, K_2$.

The full information control optimization yields the Riccati equation

$$P^2 - 2aP - 1 = 0, \quad \text{i.e. } P_{\text{stab}} = a + (a^2 + 1)^{1/2}.$$

The corresponding controller gain is

$$K_1 = -(a + (a^2 + 1)^{1/2}).$$

The state estimation optimization turns out to be the same abstract H2 optimization problem, hence $K_2 = K_1$. The resulting optimal controller is

$$K(s) = \frac{(a + (a^2 + 1)^{1/2})^2}{s + a + 2(a^2 + 1)^{1/2}}.$$

**Problem Q2S.2**

For the canonical LTI feedback setup

$$\dot{x} = u + w_1, \quad z = \begin{bmatrix} x \\ u \end{bmatrix}, \quad y = 2x + w_2, \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

find the controller which guarantees that the H-Infinity norm of the closed loop transfer matrix from $w$ to $z$ is not more than 10 percent above its lower bound.

This a the standard simplified setup for H-Infinity optimization with

$$A = 0, \quad B_1 = B_2 = 1, \quad C_1 = 1, \quad C_2 = 2.$$
For a stabilizing controller which pushes the close loop H-Infinity norm below a given threshold $\gamma > 0$ to exist, the Riccati equations

$$AP + PA' + \gamma^{-2}B_1B'_1 - B_2B'_2 + PC'_1C_1 P = 0,$$

$$QA + A'Q + \gamma^{-2}C'_1C_1 - C'_2C_2 + QB_1B'_1Q = 0,$$

must have stabilizing solutions $P = P'$, $Q = Q'$ satisfying the coupling condition

$$\begin{bmatrix}
P & \gamma^{-1}I \\
\gamma^{-1}I & Q
\end{bmatrix} > 0,$$

in which case a suboptimal controller is given by

$$u = -B'_2P^{-1}\dot{x}, \quad \dot{x} = (A + \gamma^{-2}B_1B'_1P^{-1})\dot{x} + B_2u + (Q - \gamma^{-2}P^{-1})^{-1}C'_2(y - C_2\dot{x}).$$

In this problem the Riccati equations are

$$P^2 = 1 - \gamma^{-2}, \quad Q^2 = 4 - \gamma^{-2},$$

hence stabilizing solutions

$$P = (1 - \gamma^{-2})^{1/2}, \quad Q = (4 - \gamma^{-2})^{1/2}$$

exist if and only if $\gamma > 1$, and the coupling condition

$$(1 - \gamma^{-2})^{1/2}(4 - \gamma^{-2})^{1/2} > \gamma^{-2}$$

is satisfied if and only if $\gamma > (5/4)^{1/2}$ (to get this, take a square of both sides and solve the resulting linear equation with respect to $\gamma^{-2}$).

Hence the infimum of the H-Infinity norm equals $(5/4)^{1/2}$. To obtain a suboptimal controller, take $\gamma = 1.05(5/4)^{1/2}$, and set

$$u = -(1 - \gamma^{-2})^{-1/2}\dot{x}, \quad \dot{x} = -(1 - \gamma^{-2})^{1/2}\dot{x} + 2((4 - \gamma^{-2})^{1/2} - \gamma^{-2}(1 - \gamma^{-2})^{-1/2})^{-1}(y - 2\dot{x}).$$

**Problem Q2S.3**

Let $G$ be the pure delay by 2 units of time system. Let $H$ be the Hankel operator defined by $G$. Find $g = Hf$ where $f(t) = u(t + 2) - u(t)$.

$G$ applied to $f$ yields $y(t) = u(t) - u(t - 2)$. Since $y(t) = 0$ for $t < 0$, conclude that $g(t) = y(t) = u(t) - u(t - 2)$.
Problem Q2S.4

$G$ is a stable LTI system. It is known that the Hankel operator defined by $G$ has singular numbers $\sigma_k = 2^{-k}$, $k = 1, 2, 3, \ldots$. Given the information provided, can you guarantee that

(a) it not possible to approximate $G$ by a third order LTI system $G_3$ such that the approximation error $\|G - G_3\|_\infty$ does not exceed 0.1;

(b) there exists a third order LTI system $G_3$ for which the approximation error $\|G - G_3\|_\infty$ does not exceed 0.1?

Since $\sigma_4 = 1/16 < 0.1$, it is not possible to claim that the approximation error $\|G - G_3\|_\infty$ cannot be less than 0.1. The answer to (a) is no.

Since $\sigma_4 + \sigma_5 + \cdots = 1/8 > 0.1$, it is not possible to claim that the approximation error $\|G - G_3\|_\infty$ can be made less than 0.1. The answer to (b) is no.

Problem Q2S.5

Uncertain transfer function $P(s)$ is defined as

$$P(s) = \frac{as}{1 + (1 + a\Delta_0(s))s + s^2}$$

where $a, \Delta_0$ are the uncertain parameters, $a$ ranges over $[1/2, 2/3]$, and $\Delta_0$ ranges over the set of all stable transfer functions with the H-Infinity norm not exceeding $1/2$. Derive a canonical uncertain model for $P$, i.e. find a stable rational transfer matrices $G_{00}, G_{01}, G_{10}, G_{11}$ and a cone $\Delta$ of complex matrices such that

$$P(s) = G_{00}(s) + G_{01}(s)\Delta(s)(I - G_{11}(s)\Delta(s))^{-1}G_{10}(s)$$

where $\Delta$ ranges over the set of all stable transfer matrices such that $\|\Delta\|_\infty \leq 1$ and $\Delta(j\omega) \in \Delta$ for all $\omega$.

Let $w_1$ and $z_1$ denote the input and output of $P$ respectively. Note that the uncertain parameter $a$ can be represented in the normalized form as $a = (7/12) + (1/12)\delta$, where $\delta \in [-1, 1]$, and the uncertain transfer function $\Delta_0(s)$ can be represented in the normalized form $\Delta_0(s) = (1/2)\Delta_1(s)$, where $\Delta_1$
ranges over the set of all stable transfer functions with \( \| \Delta_1 \|_\infty \leq 1 \). The system equation then takes the form

\[
\ddot{z}_1 + \dot{z}_1 + \frac{7}{12} \frac{d}{dt} (\Delta_1 z_1) + z_1 - \frac{7}{12} \dot{w}_1 + \frac{1}{12} \frac{d}{dt} (\delta (\Delta_1 z_1 - w_1)) = 0.
\]

After introducing the uncertainty representation signals

\[
w_2 = \Delta_1 z_1, \quad z_2 = z_1, \quad w_3 = \delta (w_2 - w_1), \quad z_3 = w_2 - w_1,
\]

system equation becomes

\[
\ddot{z}_1 + \dot{z}_1 + z_1 + \frac{7}{12} \dot{w}_2 - \frac{7}{12} \dot{w}_1 + \frac{1}{12} \dot{w}_3 = 0.
\]

A state-space realization of this equation can be obtained with the states

\[
x_1 = z_1, \quad x_2 = \dot{z}_1 + \frac{7}{12} w_2 - \frac{7}{12} w_1 + \frac{1}{12} w_3,
\]

which yields

\[
\dot{x}_1 = x_2 + \frac{7}{12} w_1 - \frac{7}{12} w_2 - \frac{1}{12} w_3,
\]

\[
\dot{x}_2 = -x_1 - x_2 - \frac{7}{12} w_1 + \frac{7}{12} w_2 + \frac{1}{12} w_3,
\]

\[
z_1 = x_1,
\]

\[
z_2 = x_1,
\]

\[
z_3 = -w_1 + w_2.
\]

The requested transfer matrices \( G_{ij}(s) \) are the corresponding blocks of the transfer matrix of this system, and

\[
\Delta = \left\{ \Delta = \begin{bmatrix} z & 0 \\ 0 & \delta \end{bmatrix} : \ z \in \mathbb{C}, \ \delta \in \mathbb{R} \right\}.
\]

**Problem Q2S.6**

*Write down a finite set of LMI’s depending on the coefficients of a 2-by-2 matrix \( M \) as parameters, so that the following conditions will be satisfied:

(a) the set of LMI’s has a solution whenever \( \hat{\mu}_\mathbb{C}(M) < 1; \)
(b) there exists at least one $M$ such that $\hat{\mu}_C(M) > 1$ but the set of LMI’s does have a solution;

(c) if the set of LMI’s has a solution then $I-M\Delta$ is invertible for all diagonal complex matrices $\Delta$ such that real and imaginary parts of their diagonal elements do not exceed $1/\sqrt{2}$ by absolute value.

Whenever two complex vectors $y, x$ are related by

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \Delta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Delta x,$$

where $\Delta$ satisfies the conditions mentioned in (c), the quadratic constraints

$$\sqrt{2}|x_i|^2 \pm 2\text{Re}(y_i\bar{x}_i) \geq 0,$$

$$\sqrt{2}|x_i|^2 \pm 2\text{Re}(jy_i\bar{x}_i) \geq 0,$$

$$|x_i|^2 - |y_i|^2 \geq 0$$

hold. Combining them with non-negative weights yields a family of quadratic constraints

$$x'Dx - y'Dy + \sqrt{2}x'Cx + 2\text{Re}(x'(A+jB)y) \geq 0,$$

where $D, A, B, C$ are diagonal matrices such that

$$D \geq 0, \ C \geq \pm A \pm B.$$

The resulting criterion of robustness is

$$M'DM - D + \sqrt{2}M'CM + M'(A+jB) + (A-jB)M < 0.$$ 

The LMI satisfies requirement (c). With $A = B = C = 0$ it yields the LMI for the standard upper bound for $\mu_C(M)$. Hence requirement (a) is satisfied. To show that (b) is satisfied as well, take $M = j1.01I_2$. Then $\hat{\mu}_C(M) = 1.01 > 1$, but the newly derived robustness LMI is satisfied with

$$D = 0, \ C = I, \ B = -I, \ A = 0.$$