Quiz 1 solutions

Problem Q1.1

For all values of parameter $a > 0$ find the H2 norm of

$$H(s) = \frac{e^a - e^{-s}}{s + a}.$$ 

The impulse response equals $e^{-a(t-1)}$ for $t \in [0, 1]$, and zero otherwise. Hence the square of the H2 norm is

$$\int_0^1 e^{-2a(t-1)} dt = \frac{1}{2a}(e^{2a} - 1).$$

The answer is

$$\|H\|_{H2} = \left(\frac{1}{2a}(e^{2a} - 1)\right)^{1/2}.$$ 

Problem Q1.2

For all values of parameter $a > 0$ find the H-Infinity norm of

$$H(s) = \begin{bmatrix} 1/(s + a) & 0 \\ 0 & s/(s + 1) \end{bmatrix}.$$ 

\footnote{Version of April 9, 2001}
H-Infinity norm of $1/(s+a)$ equals $1/a$ for $a > 0$ (achieved at $\omega = 0$), and equals $\infty$ for $a \leq 0$. H-Infinity norm of $s/(s+1)$ equals 1 (achieved at $\omega = \infty$). Hence

$$\|H\|_{\infty} = \begin{cases} \infty & \text{for} \quad a \leq 0 \\
\max\{1/a, 1\} & \text{for} \quad a > 0. \end{cases}$$

**Problem Q1.3**

An LTI controller $u = Ky$, where $K = K(s)$ is a proper transfer function, stabilizes the system

$$\dot{x}_1 = x_2 + w, \quad \dot{x}_2 = -x_1 + u, \quad z = u, \quad y = x_1.$$ 

Let $H = H(s)$ be the closed loop transfer function from $w$ to $z$.

(A) What are the complex frequencies $s \in \mathbb{C} \cup \{\infty\}$ at which the value of $H(s)$ does not depend on the particular selection of the stabilizing controller $K = K(s)$?

(B) Is this setup (considered for optimization of the H2 norm of $H$) singular? At which frequencies?

In this setup,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_{12} = 0, \quad D_{11} = D_{21} = 0.$$ 

Unstable control loop zeros are the zeros of

$$\begin{bmatrix} A - sI & B_1 \\ C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} -s & 1 & 1 \\ -1 & -s & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

i.e. are $s = \pm j$. Unstable sensor loop zeros are the zeros of

$$\begin{bmatrix} A - sI & B_2 \\ C_2 & D_{21} \end{bmatrix} = \begin{bmatrix} -s & 1 & 1 \\ -1 & -s & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

i.e. are 0 and $\infty$.

Hence the closed loop transfer matrix has fixed values at $s = 0, \pm j, \infty$. The setup is singular at $\omega = 1$ (a control singularity) and at $\omega = 0, \infty$ (sensor singularities).
Problem Q1.4

For which values of $a, b \in \mathbb{R}$ does the Riccati equation

$$
P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -a \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -a \end{bmatrix} P - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = P \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P
$$

have a stabilizing solution $P = P'$?

An equivalent abstract H2 optimization setup has

$$
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -a \end{bmatrix}, \quad B = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix},
$$

$$
\sigma(x, u) = bu_1^2 + u_2^2 - x_1^2 - x_2^2.
$$

The corresponding transfer function from $u_2$ to $x_2$ is $1/(s^2 + as + 1)$, which always has H-Infinity norm at least as large as 1. Hence the equivalent frequency inequality $\Pi(j\omega) > 0$ cannot be satisfied. Hence for all values of $a, b$ there exists no stabilizing solution.

Problem Q1.5

Write a system of LMI's with respect to $d_1, d_2, d_3, d_4$, and possibly some auxiliary parameters, which will be equivalent to

$$
1 - \left| \text{Re} \frac{d_1 + d_2 j\omega}{-\omega^2 + j\omega + 1} \right| - \left| \frac{d_3 + d_4 j\omega}{-\omega^2 + j\omega + 2} \right|^2 > 0 \quad \forall \ \omega.
$$

The denominators can be represented in the form

$$
-\omega^2 + j\omega + 1 = s^2 + s + 1, \quad -\omega^2 + j\omega + 2 = s^2 + s + 2,
$$

where $s = j\omega$. Consider a state space model which represents the dynamics of $1/(s^2 + s + 1)$ and $1/(s^2 + s + 2)$:

$$
\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},
$$
where the second control input is reserved for handling the square of the 
\((d_3, d_4)\)-term. A quadratic form representing the frequency domain inequality is

\[
\sigma(x, u) = u_1^2 \pm u_1(d_1x_1 + d_2x_2) + 2u_2(d_3x_3 + d_4x_4) + u_2^2.
\]

Let

\[
F = \begin{bmatrix}
0.5d_1 & 0 \\
0.5d_2 & 0 \\
0 & d_3 \\
0 & d_4
\end{bmatrix}.
\]

The equivalent system of LMI is

\[
\begin{bmatrix}
P_1A + A'P_1 & P_1B + F \\
B'P_1 + F' & I
\end{bmatrix} > 0,
\]

\[
\begin{bmatrix}
P_2A + A'P_2 & P_2B - F \\
B'P_2 - F' & I
\end{bmatrix} > 0,
\]

where \(P_1 = P'_1\), \(P_2 = P'_2\) are the auxiliary parameters.