Perfect Conductivity   Lecture 2

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Outline

1. Persistent Currents
2. Parts of a Physical Theory
3. Circuits and Time Constants
4. Distributive Systems and Time constants
   A. Quasistatics
   B. MagnetoQuasiStatics (MQS)
Persistent Currents

If the field is turned off, then

\[ I(t) = I_{t=0} \ e^{-t/\tau_{LR}} \]

The time constant \( \tau_{LR} = L/R \)

If the loop is made out of a superconductor,

\[ \lim_{R \to 0} \tau_{LR} \to \infty \quad I(t) = I_{t=0} \quad \text{for } t \geq 0. \]

Experimentally the dc resistivity of a superconductor is at least as small as \( 10^{-25} \Omega \cdot \text{m} \). The superconducting state is “truly” zero dc resistance.
Perfect Conductivity: \[ t << \tau_{RL} = \frac{L}{R}, \] system looks like \( R \) is zero

Superconductivity: for all time, \( R \) is zero
Charging up a superconducting loop

This Persistent Mode is the basis of MRI magnets, SMES, flux memory....
Parts of a Physical Theory

1. Governing Laws:
   Maxwell’s Equations, Newton’s equations,

2. Constitutive Laws:
   Models of the system
   like ohm’s law,

3. Summary Relations:
   Transfer functions, Dispersion relations
1. Governing Equations

Current conservation: \( i = i_C + i_L \), \( i_L = i_R \)

Energy Conservation: \( v = v_C = v_R + v_L \)
2. Constitutive Relations

For the resistor

\[ v_R = i_R R , \]

for the inductor as

\[ v_L = L \frac{d}{dt} i_L , \]

and for the capacitor as

\[ i_C = C \frac{d}{dt} v_C , \]

and

\[ \hat{i} \equiv |i| e^{j\phi} \]

\[ \hat{v}_R = \hat{i}_R , \]

so that

\[ v = \text{Re} \{ \hat{v} e^{j\omega t} \} \]

\[ \hat{v}_L = j\omega L \hat{i}_L \]

\[ \hat{i}_C = j\omega C \hat{v}_C \]
3. Summary Relation

\[ Z(\omega) = R \left( \frac{1 + j\omega \tau_{RL}}{1 - (\omega \tau_{LC})^2 + j\omega \tau_{RC}} \right). \]

the inductive time constant:

\[ \tau_{RL} \equiv \frac{L}{R}, \quad (1) \]

the capacitive time constant:

\[ \tau_{RC} \equiv RC, \quad (2) \]

and the coupling time constant:

\[ \tau_{LC} \equiv \sqrt{LC} = \sqrt{\tau_{RL} \tau_{RC}}. \quad (3) \]
Simpler Circuits and Time Constants

\[ \tau_{RL} \equiv \frac{L}{R} \]
Energy stored in inductor

\[ \tau_{LC} \equiv \sqrt{LC} \]
Resonant transfer of energy between L and C

\[ \tau_{RC} \equiv RC \]
Energy stored in capacitor
Reducing the Circuit to a simpler form?
Order of time constants

\[ \frac{1}{\tau_{LR}} \quad \frac{1}{\tau_{LC}} \quad \frac{1}{\tau_{RC}} \]

\[ \tau_{LR} > \tau_{RC} \quad \text{Low R} \quad R < \sqrt{\frac{L}{C}} \]

\[ \lim_{\omega \tau_{LC} \ll 1, \tau_{RC} < \tau_{RL}} Z(\omega) \approx R \left(1 + j\omega \tau_{RL}\right) \]

Low frequency circuit

\[ \tau_{LR} < \tau_{RC} \quad \text{High R} \quad R > \sqrt{\frac{L}{C}} \]

\[ \lim_{\omega \tau_{LC} \ll 1, \tau_{RL} < \tau_{RC}} Z(\omega) \approx R \left(\frac{1}{1 + j\omega \tau_{RC}}\right) \]

Low frequency circuit
Moral of time constants

If you know what frequency range you want to study or what physics dominates the problem,
then you can solve a simpler problem.

Useful, especially in more complex situations.
Distributed Systems

1. Governing Equations: Maxwell’s Equations

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday’s Law} \]

\[ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \quad \text{Ampere’s Law} \]

\[ \nabla \cdot \mathbf{D} = \rho \quad \text{Gauss Law} \]

\[ \nabla \cdot \mathbf{B} = 0 \quad \text{Gauss’ Magnetic Law} \]

Conservations laws

\[ \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad \text{Charge conservation} \quad \text{Also Poynting’s} \]
Distributed systems con’t

2. Constitutive Relations

\[ B(r, \omega) = \mu(\omega) H(r, \omega) \]

\[ D(r, \omega) = \epsilon(\omega) E(r, \omega) \quad \text{Local in space,} \]

\[ J(r, \omega) = \sigma(\omega) E(r, \omega) \quad \text{linear time invariant} \]

\[ \rightarrow \quad \text{Ohm’s Law} \]

3. Summary relations

Complex: Search first for first order in time approximation
Quasistatic Limit

\[ \ell \ll \lambda_{em} = \frac{2\pi c}{\omega} \]

Length scale of system
Wavelength of E&M wave

If the dimensions of a structure are much less than the wavelength of an electromagnetic field interacting with it, the coupling between the associated electric and magnetic fields is weak and a quasistatic approximation is appropriate.
Time Constants

\[ \tau_{em} \equiv \frac{\ell}{c} = \ell \sqrt{\mu \varepsilon} \]

Electromagnetic coupling time

\[ \tau_e \equiv \frac{\varepsilon}{\sigma_o} \]

Charge relaxation time

\[ \tau_m \equiv \mu \sigma_o \ell^2 \]

Magnetic diffusion time

\[ \tau_{em} = \sqrt{\tau_e \tau_m} \]
Order of time constants

\[ \frac{1}{\tau_m} \quad \frac{1}{\tau_{em}} \quad \frac{1}{\tau_c} \]

- \( \tau_m > \tau_c \) \quad High conductivity
- \( \sigma_o > \frac{1}{\ell} \sqrt{\frac{\epsilon}{\mu}} \)

MQS

Low frequency circuit

\[ \begin{align*}
R & \quad L \\
\text{a} & \quad \text{b}
\end{align*} \]

\[ \frac{1}{\tau_c} \quad \frac{1}{\tau_{em}} \quad \frac{1}{\tau_m} \]

- \( \tau_m < \tau_e \) \quad Low conductivity
- \( \sigma_o < \frac{1}{\ell} \sqrt{\frac{\epsilon}{\mu}} \)

EQS

Low frequency circuit

\[ \begin{align*}
C & \quad R \\
\text{a} & \quad \text{b}
\end{align*} \]
MagnetoQuasiStatics

\[ \nabla \times \mathbf{H} = \mathbf{J} \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \cdot \mathbf{J} = 0 \]

\( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \)  \( \text{Solve for } \mathbf{E} \text{ once } \mathbf{B} \text{ is found} \)

Boundary conditions:
\[ n \times (\mathbf{H}_2 - \mathbf{H}_1) = K \quad n \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \quad n \cdot (\mathbf{J}_2 - \mathbf{J}_1) = 0 \]
MQS: Magnetic Diffusion Equation

For a metal $\mathbf{B} = \mu_0 \mathbf{H}$, $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{J} = \sigma_0 \mathbf{E}$, so that

$$
\left( \mu \sigma_0 \frac{\partial}{\partial t} - \nabla^2 \right) \mathbf{H} = 0
$$

Magnetic Diffusion Equation