Lecture 7: Transmission Lines

Outline

1. Ladder Network Approximation
2. Inductance
3. Superconducting Transmission Line
4. Comparison with normal transmission line

September 25, 2003
Transmission Line: circuit model

\[
\hat{v}(x) - \hat{v}(x + \Delta x) = (j\omega L_o + R_o) \Delta x \hat{i}(x)
\]

\[
\hat{i}(x) - \hat{i}(x + \Delta x) = j\omega C_o \Delta x \hat{v}(x + \Delta x)
\]
Transmission Line

\[
\frac{d\hat{v}}{dx} = -(j\omega L_o + R_o) \hat{i} \quad \frac{d\hat{i}}{dx} = -j\omega C_o \hat{v}
\]

A wave equation is obtained

\[
\frac{d^2\hat{v}}{dx^2} = -\left(\omega^2 L_o C_o - j\omega R_o C_o\right) \hat{v}
\]

Which has solutions of the form \( \hat{v}(x) = \hat{V} e^{-j k_o x} \) with

\[
k_o = \omega \sqrt{L_o C_o} \sqrt{1 - j(R_o/\omega L_o)}
\]
Transmission line parameters

In the limit where the inductive impedance dominates,

\[
\lim_{\omega \tau_{LR} \gg 1} k_o = \omega \sqrt{L_o C_o} - j \frac{1}{2} \frac{R_o}{\sqrt{L_o/C_o}}
\]

So that \( v(x, t) = \text{Re} \left\{ \hat{V} e^{-\alpha x} e^{j \omega (t - (x/u_p))} \right\} \)

<table>
<thead>
<tr>
<th>TEM Waveguide Characteristics</th>
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</thead>
<tbody>
<tr>
<td><strong>Symbol</strong></td>
<td><strong>Name</strong></td>
</tr>
<tr>
<td>( u_p )</td>
<td>Phase Velocity</td>
</tr>
<tr>
<td>( 2\alpha )</td>
<td>Power Attenuation per Unit Length</td>
</tr>
<tr>
<td>( Z_o )</td>
<td>Characteristic Impedance</td>
</tr>
</tbody>
</table>
Fields in the Transmission Line

\[ \hat{H}_y = \frac{i}{d} \frac{\sinh \left( b - z + h/2 \right) / \lambda}{\sinh b / \lambda} \quad \text{for } 0 \leq z - (h/2) \leq b \]

\[ \hat{H}_y = \frac{i}{d} \quad \text{for } |z| \leq h/2 \]

\[ \hat{H}_y = \frac{i}{d} \frac{\sinh \left( b + z + h/2 \right) / \lambda}{\sinh b / \lambda} \quad \text{for } -b \leq z + (h/2) \leq 0 \]
Fields in the Transmission Line

\[ \hat{H}_y = \frac{\hat{i}}{d} \frac{\sinh (b_1 - z + h/2)/\lambda_1}{\sinh b_1/\lambda_1} \quad \text{for} \quad 0 \leq z - (h/2) \leq b_1 \]

\[ \hat{H}_y = \frac{\hat{i}}{d} \quad \text{for} \quad |z| \leq h/2 \]

\[ \hat{H}_y = \frac{\hat{i}}{d} \frac{\sinh (b_2 + z + h/2)/\lambda_2}{\sinh b_2/\lambda_2} \quad \text{for} \quad -b \leq z + (h/2) \leq 0 \]
Bulk Superconducting Transmission Line

\[ C_0 = \varepsilon_t \frac{d}{h} \]

\[
\lim_{\lambda \ll \delta} R_o = \frac{4}{d\delta} \left( \frac{\lambda}{\delta} \right)^3 = \frac{2}{d} \text{Re} \{Z_S\}
\]

\[
\lim_{\lambda \ll \delta} L_o = \mu_t \frac{h}{d} + 2\mu_o \frac{\lambda}{d}
\]

\[
\lim_{\lambda \ll \delta} L_o = \frac{2}{d} \text{Im} \{Z_S\}
\]

Two identical, thick (\(\lambda \ll b\)) superconducting plates.
Inductance

Inductance per unit length is found from

$$\frac{1}{4} \int dy \int dz \left( \mu |\hat{H}|^2 + \Lambda \left| \hat{J}_s \right|^2 \right) = \frac{1}{4} L_o \left| \hat{i} \right|^2,$$

Inside the transmission line space

$$\hat{H}_y = \frac{\hat{i}}{d} \quad \text{for} \quad |z| \leq h/2$$

$$L_{o,in} = \mu \left(\frac{1}{d}\right)^2 dh = \mu \frac{h}{d}$$

Inside the transmission line material

$$\hat{H}_y = \frac{\hat{i}}{d} \frac{\sinh k (b - z + (h/2))}{\sinh kb} \quad \text{for} \quad 0 \leq z - (h/2) \leq b$$

$$\lim_{\lambda \ll \delta} \lim_{\lambda \ll b} L_{o,material} = 2L_s = 2 \mu_o \frac{\lambda}{d}$$
Inductance for a thin slab

The current density is uniform for the slab so that

\[ \mathbf{J} = \frac{\hat{i}}{bd} \mathbf{i}_x \]

The energy stored in the slab is

\[ W = \frac{1}{2} \mu_0 \lambda^2 (\mathbf{J})^2 b = \mu_0 \lambda^2 \left( \frac{\hat{i}}{bd} \right)^2 b d \Delta x = \frac{1}{2} L_0 \Delta x \hat{i}^2 \]

Therefore, \[ L_0 = \frac{\mu_0 \lambda^2}{db} \]

For each slab and the total inductance per unit length is twice this. This is the kinetic inductance.
Dispersionless Transmission Lines

Because $L_o$ and $C_o$ do not depend on frequency for a superconductor, the phase velocity is independent of frequency. So that a pulse will propagate down a superconducting transmission line without dispersing. Also, the amount of attenuation is extremely small, since this is due to $R_o$.

For a normal metal, $L_o$ depends on frequency so that there is dispersion, in addition to a much greater loss.
## Summary

<table>
<thead>
<tr>
<th>Transmission Line Geometry</th>
<th>$L_o$</th>
<th>$C_o$</th>
<th>$R_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two identical, thin ($\lambda \gg b$) superconducting plates.</td>
<td>$\frac{\mu_r h}{d} + \frac{2\mu_0 \lambda^2}{db}$</td>
<td>$\frac{\varepsilon_r d}{h}$</td>
<td>$\frac{8}{db\sigma_p} \left(\frac{\lambda}{\delta}\right)^4$</td>
</tr>
<tr>
<td>Two identical, thick ($\lambda \ll b$) superconducting plates.</td>
<td>$\frac{\mu_r h}{d} + \frac{2\mu_0 \lambda}{d}$</td>
<td>$\frac{\varepsilon_r d}{h}$</td>
<td>$\frac{4}{d\delta\sigma_o} \left(\frac{\lambda}{\delta}\right)^3$</td>
</tr>
<tr>
<td>One thick ($\lambda \ll b$) superconducting plate and one thick ($\lambda \ll b$) ohmic plate.</td>
<td>$\frac{\mu_r h}{d} + \frac{\mu_0 \lambda}{d} + \frac{\mu_n \delta_n}{2d}$</td>
<td>$\frac{\varepsilon_r d}{h}$</td>
<td>$\frac{1}{d\delta_n \sigma_{o,n}}$</td>
</tr>
</tbody>
</table>