Exam 2: Equation Summary

Newton’s Second Law: Force, Mass, Acceleration: \( \vec{F} = m \vec{a} \)

Newton’s Third Law: \( \vec{F}_{2,1} = -\vec{F}_{1,2} \)

Center of Mass: \( \vec{R}_{cm} = \frac{1}{m_{total}} \sum_{i=1}^{N} m_i \vec{r}_i \rightarrow \frac{1}{m_{total \, body}} \int dm \vec{r} \)

Velocity of Center of Mass: \( \vec{V}_{cm} = \frac{1}{m} \sum_{i=1}^{N} m_i \vec{v}_i \rightarrow \frac{1}{m_{total \, body}} \int dm \vec{v} \)

Momentum: \( \vec{p} = m \vec{v}, \quad \vec{p}_{sys} = \sum_{i=1}^{N} m_i \vec{v}_i \)

Newton’s Second Law \( \vec{F}_{sys} = \frac{d\vec{p}_{sys}}{dt} \)

Impulse: \( \vec{I} \equiv \int_{t=0}^{t=\tau} \vec{F}(t) dt = \Delta \vec{p} \)

Kinetic Energy: \( K = \frac{1}{2} mv^2; \quad \Delta K \equiv \frac{1}{2} mv_f^2 - \frac{1}{2} mv_0^2 \)

Work- Kinetic Energy: \( W = \int_{A}^{B} \vec{F} \cdot d\vec{r} \quad W = \Delta K \)
Problem 1 of 4 (25 points): In this problem you must show your work. Equations without justification will receive no credit.

A spaceship of dry mass $m_0$, carrying fuel of mass $m_f$, is traveling in interstellar space (zero gravity) and is initially moving with speed $v_0$ to the right. The spaceship would like to dock with a space station that is moving to the right with speed $v_f < v_0$. The spaceship begins to slow down by ejecting burned fuel in the forward direction at a constant speed $u$ relative to the spaceship. (At the instant depicted in the figure the spaceship has speed $v$ to the right.)

a) Derive a relation between the differential of the speed of the spaceship $dv$, and the differential of its total mass $dm$.

b) What is the engine exhaust speed $u$ such that the final speed of the spaceship is the same as the space station after all the fuel has been burned? Express your answer in terms of $m_0$, $m_f$, $v_0$, and $v_f$ as needed.
Problem 2 of 4 (25 points):

Block A of mass $m_A$ is moving horizontally with speed $V_A$ along a frictionless surface. It collides with block B of mass $m_B$ that is initially at rest. The two blocks stick together after the collision. At $x = 0$, block B enters a rough surface at with a coefficient of kinetic friction that increases linearly with distance $\mu_k(x) = bx$ for $0 \leq x \leq d$, where $b$ is a positive constant. At $x = d$ block B collides with an unstretched spring with spring constant $k$ on a frictionless surface. The downward gravitational acceleration has magnitude $g$.

What is the distance the spring is compressed when the blocks first comes to rest? Express your answer in terms of $V_A$, $m_A$, $m_B$, $b$, $d$, $g$, and $k$.

Solution

Because there are no external forces in the $x$-direction, momentum is constant during the collision hence

$$m_A V_A = (m_A + m_B)v_1.$$ 

So the speed immediately after the collision is

$$v_1 = \frac{m_A}{m_A + m_B} V_A.$$
\[ W = \int_{0}^{\infty} F_f \cdot d\bar{r} + \int_{0}^{x_f} F_s \cdot d\bar{r} \]
\[ = -\int_{0}^{\infty} b x (m_A + m_B) g \, dx - \int_{0}^{x_f} k x \, dx \]
\[ = -b (m_A + m_B) g \frac{d^2}{2} - \frac{1}{2} k x_f^2 \]
\[ W = \Delta K \implies \]
\[ -b (m_A + m_B) \frac{d^2}{2} - \frac{1}{2} k x_f^2 = 0 - \frac{1}{2} (m_A + m_B) v_1^2 \]
\[ \implies \]
\[ x_f = \sqrt{\frac{(m_A + m_B) (v_1^2 - bg \, dx^2)}{k}} \]

Use Eq. (3) \[ \implies \]
\[ x_f = \sqrt{\left( \frac{1}{k} \left( \frac{(m_A V_A)^2}{m_A + m_B} - bg \, dx^2 \right) \right)} \]
Problem 3 of 4 (25 points):

A uniform rope of mass $m_1$ and length $l$ is attached to shaft that is rotating at constant angular velocity $\omega$. The rope has a point-like object of mass $m_2$. Find the tension in the rope as a function of distance from the shaft. You may ignore the effect of gravitation.

Solution:

Divide the rope into small pieces of length $\Delta r$, each of mass $\Delta m = (m_1 / l) \Delta r$.

Consider the piece located a distance $r$ from the shaft. The radial component of the force on that piece is the difference between the tensions evaluated at the sides of the piece, $F_r = T(r + \Delta r) - T(r)$.

The piece is accelerating inward with a radial component $a_r = -r \omega^2$. Thus Newton’s Second Law becomes
\[ F_r = -\Delta m \omega^2 r \]
\[ T(r + \Delta r) - T(r) = -(m_1 / l) \Delta r \omega^2. \] (1)

Denote the difference in the tension by \( \Delta T = T(r + \Delta r) - T(r) \). After dividing through by \( \Delta r \) Eq. (1) becomes

\[ \frac{\Delta T}{\Delta r} = -(m_1 / l) \omega^2. \] (2)

In the limit as \( \Delta r \to 0 \), Eq. (2) becomes a differential equation,

\[ \frac{dT}{dr} = -(m_1 / l) \omega^2 r. \] (3)

From this, we see immediately that the tension decreases with increasing radius. We shall solve this equation by integration

\[ T(r) - T(l) = \int_{r'}^{r''} \frac{dT}{dr'} dr' = -\left( m \omega^2 / l \right) \int_{r}^{\infty} r' dr' = -(m \omega^2 / 2l)(r^2 - l^2). \] (4)

Thus the tension \( T(r) \) in the rope as a function of the distance \( r \) form the center of the rotating shaft

\[ T(r) = (m \omega^2 / 2l)(l^2 - r^2) + T(l) \] (5)

In order to determine the tension \( T(l) \) at the end of the rope consider the free body force diagram on the object of mass \( m_2 \) at the end of the rope.

The magnitude of the force on the object due to the rope is equal to the tension \( T(l) \) at the end of the rope. Newton’s Second Law becomes

\[ -T(l) = -m_2 l \omega^2 \] (6)
The tension at the end of the rope is thus

\[ T(l) = m_2 l \omega^2 \]  \hfill (7)

Substituting Eq. (7) into Eq. (4) yields an expression for the tension \( T(r) \) in the rope as a function of the distance \( r \) from the center of the rotating shaft

\[ T(r) = \left( \frac{m_1 \omega^2}{2l} \right) (l^2 - r^2) + m_2 l \omega^2 \]  \hfill (8)
Problem 4 of 4: (25 points) Concept Questions

Part A (5 points):

Cart A is pushed continuously from the start line to the finish line with a force $F_A$.
Cart B is pushed continuously from the start line to the finish line with twice the force, $F_B = 2F_A$. Cart A has twice the mass of cart B, $m_A = 2m_B$. Both carts are initially at rest.

a) Cart A has a larger change in momentum and a larger change in kinetic energy than cart B.

b) Cart A has a larger change in momentum but equal change in kinetic energy than cart B.

c) Cart A has a larger change in momentum but smaller change in kinetic energy than cart B.

d) Cart A has an equal change in momentum but larger change in kinetic energy than cart B.

e) Both carts have the equal change in momentum and equal change in kinetic energy.

f) Cart A has an equal change in momentum but smaller change in kinetic energy than cart B.

gh) Cart A has a smaller change in momentum but larger change in kinetic energy than cart B.

h) Cart A has a smaller change in momentum and equal change in kinetic energy than cart B.

i) Cart A has a smaller change in momentum and smaller change in kinetic energy than cart B.
Answer: f)

\[ W_A = F_A d = k_A \]
\[ W_B = F_B d = 2F_A d = k_B \]

\[ \Rightarrow 2k_A = k_B \Rightarrow k_A = \frac{k_B}{2} \]

Cart A has a smaller change in kinetic energy than cart B

\[ k_A = \frac{P_A^2}{2m_A} = \frac{P_A^2}{2(2m_B)} = \frac{1}{2} \frac{P_B^2}{2m_B} \Rightarrow \]

\[ P_A = P_B \]

Cart A has an equal change in momentum as cart B.
Part B (5 points):

A system is composed of two non-identical blocks connected by a spring. The blocks slide on a frictionless plane. The unstretched length of the spring is $d$. Initially block 2 is held so that the spring is compressed to $d/2$ and block 1 is forced against a stop as shown in the figure above. Block 2 is released. Which of the following statements is true? (Note: more than one statement may be true.)

a) When the position of block 2 is $x_2 > d$, the center of mass of the system is accelerating to the right.

b) When the position of block 2 is $x_2 > d$, the center of mass of the system is moving at a constant speed to the right.

c) When the position of block 2 is $x_2 > d$, the center of mass of the system is at rest.

d) When the position of block 2 is $x_2 < d$, the center of mass of the system is accelerating to the right.

e) When the position of block 2 is $x_2 < d$, the center of mass of the system is moving at a constant speed to the right.

f) When the position of block 2 is $x_2 < d$, the center of mass is at rest.

Enter the letters of all correct statements on the answer line below.
Answer: b) and d)

When $x_2 < d$, the wall exerts an external force on block 1, to the right, hence the center of mass of the system consisting of block 1 and block 2 is accelerating to the right.

When the block 2 reaches $x_2 = d$, the center of mass is moving to the right. The spring begins to exert a force on block 1 and block 1 leaves the wall. After that moment, there is no external force acting on the system consisting of block 1 and block 2 so the center of mass moves at a constant speed to the right.
A small object of mass $m$ is hanging vertically from a string suspended from a fixed point on the ceiling. The object is pulled to the left through an angle $\theta$ and released from rest, undergoing motion along a circular arc. When the object is at the bottom of its trajectory (point A in figure), which of the following statements is true? (Note: more than one statement may be true.)

a) The tension in the string is greater than the magnitude of the gravitational force acting on the object.

b) The tension in the string is equal to the magnitude of the gravitational force acting on the object.

c) The tension in the string is less than the magnitude of the gravitational force acting on the object.

d) The component of the acceleration in the direction of motion is positive.

e) The component of the acceleration in the direction of motion is negative.

f) The component of the acceleration in the direction of motion is zero.
Answer: a) and f)

When the object is at the bottom of the circular arc,

\[ F_r = ma_r \]

\[ mg - T = -\frac{mv^2}{r} \]

\[ T = mg + \frac{mv^2}{r} \]

The tension is greater than the magnitude of the gravitational force.

Because \( F_\theta = 0 \) \( \Rightarrow \) \( r \dot{\theta} = 0 \).

The component of acceleration in the direction of motion is zero.
A ring lies on its side on a frictionless table. It is pivoted to the table at its rim. A bug walks on the ring with constant speed relative to the ring. Which of the following statements is true?

a) The momentum of the bug and ring is constant.

b) The momentum of the bug and ring is not constant.

c) The work done by the pivot force is zero.

d) The work done by the pivot force is non-zero.

Enter the letters of all correct statements on the answer line below.

Answer: b) and c)

The pivot force is a non-zero external force on the system so the momentum of the bug and ring is non-constant. The pivot point is not displaced, so the work done by the pivot force is zero.
Part E (5 points):

Suppose you drop paper clips vertically down into an open cart rolling along a straight horizontal track with negligible friction. Assume the speed of the falling paper clips is much less than the speed of the cart.

As a result of the accumulating paper clips in the cart, the cart’s

1) kinetic energy increases and magnitude of the momentum increases.
2) kinetic energy increases and the magnitude of the momentum stays the same.
3) kinetic energy increases and the magnitude of the momentum decreases.
4) kinetic energy stays the same and the magnitude of the momentum increases.
5) kinetic energy stays the same and magnitude of the momentum stays the same.
6) kinetic energy stays the same and the magnitude of the momentum decreases.
7) kinetic energy decreases and the magnitude of the momentum increases.
8) kinetic energy decreases and the magnitude of the momentum stays the same.
9) kinetic energy decreases and magnitude of the momentum decreases.

Enter the number of the correct statement on the answer line below.
Answer: 9.

The cart exerts a force on the paper clips increasing the paper clips speed in the horizontal direction and decreasing the speed in the vertical direction. If we choose just the cart as the system, then the paper clips exert and equal and opposite force on the cart. The horizontal component of that force slows the cart down. So the kinetic energy of the cart decreases.

Consider the momentum diagrams (positive $x$-direction chosen to the right) for the system of the cart of mass $m_c(t)$ at time $t$ when it is traveling with an $x$-component of velocity $v_c(t)$ and a small amount of paper clips of mass $\Delta m_p$ entering the cart with a vertical speed $u$. There are no external forces in the $x$-direction so

$$\Delta m_p v + m_c \Delta v = 0$$

$$\Rightarrow m_c \Delta v = -\Delta m_p v \Rightarrow m_c \Delta v < 0$$

The momentum of the cart decreases.