A small toy is suspended a height $h$ above the ground. At time $t = 0$, the toy is dropped and experiences a time-varying acceleration due to the gravitational force and air resistance, with the $y$-component of the acceleration given by $a_y(t) = -g + bt^2$, where $b$ is a positive constant and $g$ is the magnitude of the gravitational acceleration. A physics instructor has set up a projectile gun a horizontal distance $d$ away from the toy. The projectile is initially a height $s$ above the ground. The instructor fires the projectile with an initial velocity of magnitude $v_0$ at an angle $\theta_0$ with respect to the horizontal. The projectile has negligible air resistance, and it hits the toy at the time $t = t_f$. Find an expression for the initial angle $\theta_0$ only in terms of the quantities $s$, $h$, $d$, $b$, $t_f$, and $g$. Do not use $v_0$ in your answer.
Problem 1

Initial conditions:
\( x(0) = y(0) = 0 \)

\( \dot{x}_1(0) = v_{x,1}(0), \quad \dot{y}_1(0) = v_{y,1}(0) = v_0 \sin \theta \)

\( \dot{x}_1(0) - x(0) = 0 \Rightarrow \dot{x}_1(0) = v_0 \cos \theta \)

\( x(t) = \int_{0}^{t} v_0 \cos \theta \, dt' = v_0 \cos \theta \, t \Rightarrow \)

\( x(t) = v_0 \cos \theta \, t \)

\( \dot{y}_1(0) - y(0) = \int_{0}^{t} -g \, dt' = -g \, t \)

\( \dot{y}_1(t) = v_0 \sin \theta - g \, t \)

\( y(t) = \int_{0}^{t} (v_0 \sin \theta - g \, t') \, dt' = v_0 \sin \theta \, t - \frac{1}{2} g \, t^2 \Rightarrow \)

\( y(t) = v_0 \sin \theta \, t - \frac{1}{2} g \, t^2 \)
Initial conditions: \( y_2(0) = h - s, \ y'_2(0) = 0 \)

\[ y_2(t) = \frac{-g}{b} t^2 \]

\[ v_{y_2}(t) = \int_0^t (g + b t^2) \, dt' \]

\[ v_{y_2}(t) = -g t + \frac{b t^3}{3} \]

\[ y_2(t) = y_2(0) - \frac{1}{2} g t^2 + \frac{1}{12} b t^4 \]

when toy ball hits toy at \( t = t_f \),

\[ y_1(t_f) = y_2(t_f), \quad x_1(t_f) = x_2(t_f) = d \]

\[ \Rightarrow \quad v_0 \sin \theta_0 \, t_f - \frac{1}{2} g \, t_f^2 = (h - s) - \frac{1}{2} g t_f^2 + \frac{1}{12} b t_f^4 \]

\[ \Rightarrow \quad v_0 \sin \theta_0 \, t_f = (h - s) + \frac{1}{12} b t_f^4 \]  \hspace{1cm} (1)

\[ v_0 \cos \theta_0 \, t_f = d \]  \hspace{1cm} (2)

divide eq (1) by eq (2) \( \Rightarrow \)

\[ \tan \theta_0 = \frac{(h - s) + \frac{1}{12} b t_f^4}{d} \]

\[ \theta_0 = \tan^{-1} \left( \frac{(h - s) + \frac{1}{12} b t_f^4}{d} \right) \]
Problem 2 of 4 (25 points)

Two blocks 1 and 2 of $m_1$ and $m_2$, respectively, with $m_1 > m_2$, are stacked on a fixed inclined plane and connected by an inextensible massless string attached to a pulley, as shown in the figure above. The pulley is massless and frictionless. There is no friction between block 1 and block 2. However, there is friction between block 1 and the inclined plane, with static coefficient of friction $\mu$. If the angle $\theta$ is gradually increased from zero, the blocks start to slip when some angle $\theta = \theta_0$, with block 1 slipping downward. The gravitational acceleration $g$ is directed downward.

a) Draw force diagrams for each block. Label the forces clearly.

b) Find the angle $\theta_0$ at which the blocks start to slip.
Problem 2  \[ \text{static: } a_1 = a_2 = 0 \]

\[ i: \quad m_1 g \sin \theta - T - (f_s)_x = 0 \quad (1) \]

\[ j: \quad N_{6,1} - N_{2,1} - m_1 g \cos \theta = 0 \]

\[ \Rightarrow N_{112} = m_2 g \cos \theta \]

at \( \theta = \theta_0 \), \( (f_s)_x = \mu N_{6,1} \) maximum value

\[ N_{6,1} = N_{112} + m_1 g \cos \theta \Rightarrow N_{6,1} = (m_1 + m_2) g \cos \theta \]

\[ \Rightarrow m_1 g \sin \theta_0 - T - \mu (m_1 + m_2) \cos \theta_0 = 0 \quad (2) \]

\[ T = m_2 g \sin \theta_0 \]

\[ (m_1 - m_2) g \sin \theta_0 = \mu (m_1 + m_2) g \cos \theta_0 \]

\[ \tan \theta_0 = \mu (m_1 + m_2) \Rightarrow \theta_0 = \tan^{-1} \left( \frac{\mu (m_1 + m_2)}{m_1 - m_2} \right) \]
Problem 3 of 4 (25 points)

In the system shown in the figure below the masses of the three blocks are \( m_1 \), \( m_2 \), and \( m_3 \). All three blocks are moving. The coordinates of the three blocks \((x_1(t), x_2(t), \text{ and } y_3(t))\) are defined as shown in the figure. Assume that the pulleys are massless and frictionless, and the inextensible rope joining the blocks is massless. The coefficient of kinetic friction between the blocks and the tables is \( \mu_k \). The gravitational acceleration \( g \) is directed downward.

a) Draw free-body force diagrams on each of the blocks. Label the forces clearly.

b) Determine an expression relating \( a_1 \equiv \frac{d^2x_1}{dt^2} \), \( a_2 \equiv \frac{d^2x_2}{dt^2} \), and \( a_3 \equiv \frac{d^2y_3}{dt^2} \).

c) Determine the tension in the string connecting block 1 to block 2 only in terms of the quantities \( m_1, m_2, m_3, \mu_k \), and \( g \).
Problem 3:

\( \ell_1 = x_1 + \left( \frac{\pi}{2} \right)^2 + 2 y_3 + \pi R + \left( \frac{\pi}{2} \right) R + x_2 \)

\( \frac{\partial^2 \ell_1}{\partial t^2} = c = a_{x_1} + 2 a_{y_3} + a_{x_2} \)

\( (f_k)_1 \rightarrow T \rightarrow \hat{\ell}_1 \rightarrow \hat{j}_1 \rightarrow \hat{\ell}_1 : (f_k)_1 = m_1 a_{x_1} \) \( (1) \)

\( \hat{j}_1 : \hat{N}_{T,1} - m_1 g = 0 \) \( (2) \)

\( (f_k)_1 = M_k \hat{N}_{T,1} = M_k m_1 g \Rightarrow \)

\( 0 \Rightarrow M_k m_1 g - T = m_1 a_{x_1} \) \( (3) \)

\( T \rightarrow \hat{\ell}_2 \rightarrow \hat{j}_2 \rightarrow \hat{\ell}_2 : (f_k)_2 = m_2 a_{x_2} \) \( (4) \)

\( \hat{j}_2 : \hat{N}_{T,2} - m_2 g = 0 \)

\( (f_k)_2 = M_k \hat{N}_{T,2} = M_k m_2 g \)

\( (3) \Rightarrow M_k m_2 g - T = m_2 a_{x_2} \) \( (5) \)
\[ T = \frac{2 g (M_k + 1)}{\left( \frac{1}{m_1} + \frac{1}{m_3} + \frac{1}{m_2} \right)} \]
Problem 4 of 4 (25 points): Concept Questions (Parts A through E)

Part A (5 points)

A person of mass $m$ is standing in an elevator that is accelerating downward with magnitude $a$. The gravitational acceleration $g$ is directed downward and $a < g$. The magnitude $N = |N|$ of the normal force exerted by the elevator on the person is

1. $N = mg$.
2. $N = mg + ma$.
3. $N = mg - ma$.
4. $N = 0$.
5. $N = ma - mg$.
6. None of the above.

Answer 3: $mg - N = ma \Rightarrow N = mg - ma$
Part B (5 points)

At $t = 0$, an object is located at the origin. The object moves along the $x$-axis. The position vector of the object as a function of time is given by

$$x(t) = -v_{x0} t + (1/3)bt^3,$$

where $b$ is a positive constant and $v_{x0} > 0$ is the $x$-component of the velocity at $t = 0$. Which of the following statements is true?

1. At time $t = 0$, the object is moving in the positive $x$-direction and at some later time, the object will stop and reverse direction.

2. At time $t = 0$, the object is moving in the negative $x$-direction and at some later time, the object will stop and reverse direction.

3. At time $t = 0$, the object is moving in the positive $x$-direction and at some later time, the object will stop and remain at rest.

4. At time $t = 0$, the object is moving in the negative $x$-direction and at some later time, the object will stop and remain at rest.

5. The $x$-component of the acceleration of the object is decreasing as a function of time.

6. The $x$-component of the acceleration of the object starts off with a negative non-zero value at $t = 0$ and then increases as a function of time.

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**Answer 2:**

$$x(t) = -v_{x0} t + (1/3)bt^3$$

$$v_x(t) = -v_{x0} + bt^2$$

$$a_x(t) = 2b t \geq 0 \text{ for } t \geq 0$$

At $t = 0$, $v_x(0) = -v_{x0} < 0$.

Object traveling in negative $x$-direction.

$$0 = v_x(t_1) = -v_{x0} + bt_1^2 \Rightarrow t_1 = \sqrt{\frac{v_{x0}}{b}}$$

Object comes to rest at $t_1$.

Then for $t > t_1$, $v_x(t) > 0$, so object reverses direction.
Part C (5 points)

In the system shown in the figure above, the pulleys are massless and frictionless, and the ropes have no mass. Choose downward as the positive direction for acceleration of each moving object. The gravitational acceleration \( g \) is directed downward. What is the relationship between the downward components of the accelerations of blocks 1 and 2 due to the constraints imposed by the system of strings and pulleys?

1. \( 0 = a_1 + a_2 \).
2. \( 0 = a_1 - a_2 \).
3. \( 0 = 2a_1 + a_2 \).
4. \( 0 = 2a_1 - a_2 \).
5. \( 0 = a_1 + 2a_2 \).
6. \( 0 = a_1 - 2a_2 \).
7. None of the above

\[ \text{Answer:} \]

\[ l_1 = y_1 + \pi R + y_p + \pi R + y_p + s \]
\[ \frac{d^2 l_1}{dt^2} = a y_1 + 2 a y_p \quad (1) \]
\[ l_2 = y_2 - y_p \]
\[ \frac{d^2 l_2}{dt^2} = a y_2 - a y_p \Rightarrow a y_2 = a y_p \]
\[ \text{Eq (1):} \quad 0 = a y_1 + 2 a y_2 \]
Part D (5 points)

A person is climbing an inextensible rope that passes over a pulley and is attached to a block at the other end. The pulley is attached to the ceiling. The person and the block have the same mass $m$. You may neglect the mass of the pulley and the rope. Assume that the person is pulling on the rope with a constant force of magnitude $F$. Denote the acceleration of the person by $\mathbf{a}_{\text{person}}$ and the acceleration of the block by $\mathbf{a}_{\text{block}}$. The gravitational acceleration $g$ is directed downward. Which of the following statements is true?

1. The person accelerates upward and $\mathbf{a}_{\text{block}} = \mathbf{0}$.

2. The person and the block both accelerate upward and $|\mathbf{a}_{\text{person}}| > |\mathbf{a}_{\text{block}}| > 0$.

3. The person and the block both accelerate upward and $|\mathbf{a}_{\text{person}}| = |\mathbf{a}_{\text{block}}|$.

4. The person and the block both accelerate upward and $|\mathbf{a}_{\text{person}}| < |\mathbf{a}_{\text{block}}|$.

5. The person accelerates upward and the block accelerates downward and $|\mathbf{a}_{\text{person}}| > |\mathbf{a}_{\text{block}}| > 0$.

6. The person accelerates upward and the block accelerates downward and $|\mathbf{a}_{\text{person}}| = |\mathbf{a}_{\text{block}}|$.

7. The person accelerates upward and the block accelerates downward and $|\mathbf{a}_{\text{person}}| < |\mathbf{a}_{\text{block}}|$.
Answer 3:

\[ T \uparrow \quad T \uparrow \quad T \uparrow \quad T \uparrow \]

\[ m g \quad m g \quad m g \quad m g \]

\[ T - m g = m a_{\text{person}} \quad T - m g = m a_{\text{block}} \]

\[ \Rightarrow a_{\text{person}} = a_{\text{block}} \]

\[ \vec{a} \text{ is upward and } |\vec{a}_{\text{person}}| = |\vec{a}_{\text{block}}| \]
Part E (5 points)

Block 1 of mass $m$ is in contact with block 2 of mass $4m$. The two blocks are on a horizontal surface with negligible friction. Suppose a horizontal force $\vec{F}_1$ is applied to the left side of block 1 and a second horizontal force $\vec{F}_2$ is applied to the right side of block 2, as shown in the figure below, with magnitudes satisfying $F_2 = |\vec{F}_2| = 2|\vec{F}_1| = 2F_1$. Let $\vec{F}_{21}$ denote the force on block 1 due to the contact with block 2. Denote the magnitude by $F_{21} = |\vec{F}_{21}|$. Let $a$ denote the magnitude of the acceleration of the blocks. Which of the following statements is true?

1. $F_1 = F_{21} = ma$.
2. $F_1 = F_{21} = 5ma$.
3. $F_1 = ma$ and $F_{21} = 2ma$.
4. $F_1 = ma$ and $F_{21} = 0$.
5. $F_1 = 5ma$ and $F_{21} = 6ma$.
6. $F_1 = 5ma$ and $F_{21} = 4ma$.
7. None of the above.
Answer 5:

\[ F_1 - F_1 = m \cdot a, \quad F_2 - F_{12} = 4m \cdot a \]

\[ q_1 = a_2 = a, \quad F_{12} = F_{21}, \quad F_2 - F_1 = m \cdot a, \quad F_2 - F_{12} = 4m \cdot a \]

\[ F_2 - F_1 = m \cdot a, \quad 2F_1 - F_{12} = 4m \cdot a \]

\[ \Rightarrow F_{21} = F_1 + ma \quad \Rightarrow 2F_1 - (F_1 + ma) = 4m a \]

\[ \Rightarrow F_1 = 5m a \]

\[ \Rightarrow F_{21} = 5ma + ma = 6ma \]

\[ \text{Note} \quad \Rightarrow \quad F_{12} = \frac{F_2 - 4m a}{2F_1} \quad \Rightarrow 2F_1 - F_1 = 5ma \quad \Rightarrow \quad F_1 = 5ma \]