Problem 1 of 4 (25 points)

Answers without work shown will not be given any credit.

A simple pendulum consists of a point-like object of mass \( m \) that is suspended from a frictionless pivot by a string of length \( L \) and negligible mass. The object is initially at rest at the bottom of the swing. At \( t = 0 \), the object is given an instantaneous impulse resulting in an initial speed \( v_i \) and swings upward in a circular orbit, as shown in the figure below left. At \( t_f \) the object has reached an angle \( \theta_f \) with respect to the vertical with a speed \( v_f \), as shown in the figure below right. The downward acceleration of gravity is \( g \). Express all answers in terms of \( m, \theta_f, g, \) and \( L \), as needed.

a) After the impulse is applied, what is the work done on the object as it moves from the bottom of the swing to an angle \( \theta_f \) from the vertical? Express your answer in terms of \( m, \theta_f, g, t_f, \) and \( L \) as needed. Do not use \( v_i \) or \( v_f \) in your answer.

\[
W = \int F \cdot dr = -mg \int \sin \theta \, d\theta
\]

\[
= -mgL(1 - \cos \theta_f)
\]

Tension does no work because \( \vec{T} \cdot d\vec{r} = 0 \)
b) What is the tension in the string when the object first reaches the angle $\theta_f$? (Note that the object is still moving with speed $v_f$.) Express your answer in terms of $m$, $\theta_f$, $g$, $t_f$, $v_i$, and $L$ as needed. Do not use $v_f$ in your answer.

\[
\begin{align*}
F_r &= ma_r \\
\vec{r} : -T + mg \cos \theta_f &= -m \frac{v_f^2}{L} & 4 \text{ points} \\
W &= \Delta K \\
-mgL(1-\cos \theta_f) &= \frac{1}{2} m \frac{v_f^2}{L} - \frac{1}{2} m \frac{v_i^2}{L} \\
\frac{m v_i^2}{L} - 2mg(1-\cos \theta_f) &= m \frac{v_f^2}{L} & 6 \text{ points} \\
T &= \frac{m v_f^2}{L} + mg \cos \theta_f & \text{correct} \\
&= \frac{m v_i^2}{L} - 2mg + 3mg \cos \theta_f & \text{answer} & 3 \text{ points}
\end{align*}
\]
In gravity-free space, a spacecraft of mass $m_1$ collects debris that lies in its path. The spacecraft is initially moving with speed $v_1$.

a) The spacecraft encounters a piece of debris of mass $m_2$ that is moving with speed $v_2$ directly toward the spacecraft, as shown above. What is the speed $v_a$ of the spacecraft after it has collected the debris?

\[
\begin{align*}
\vec{p}_i &= \vec{p}_f \\
m_1v_1 - m_2v_2 &= (m_1 + m_2)v_a \\
\Rightarrow v_a &= \frac{m_1v_1 - m_2v_2}{m_1 + m_2}
\end{align*}
\]
b) After collecting the piece of debris, the spacecraft fires its engine by ejecting fuel exhaust backwards at speed \( u \) relative to the spacecraft. Derive a relation between the differential of the speed of the spacecraft, \( dv \), and the differential of the mass of the spacecraft, \( dm \), when the spacecraft has total mass \( m \) and is traveling at speed \( v \).

(Note that the total mass includes the dry mass of the spacecraft, the fuel on board, and the debris collected.)
c) The engine continues to run until the spacecraft again reaches the original speed $v_1$ that it had before its encounter with the debris. The amount of fuel burned was exactly equal to the mass of the piece of debris $m_2$. Find an expression for the exhaust speed $u$ (relative to the spacecraft) of the ejected fuel. Express your answer in terms of $m_1$, $m_2$, $v_1$, and $v_a$, as needed.

\[
\begin{align*}
\vec{v}' &= \vec{v}_1 \\
\int d\vec{v}' &= -u \int \frac{dm'}{m'} \\
\vec{v}' &= \vec{v}_a \\
\vec{v}_1 - \vec{v}_a &= -u \ln \left( \frac{m_1}{m_1 + m_2} \right) \\
\mathbf{u} &= \frac{\vec{v}_1 - \vec{v}_a}{\ln \left( \frac{m_1 + m_2}{m_1} \right)}
\end{align*}
\]
Problem 3 of 4 (25 points)

Answers without work shown will not be given any credit.

Object 1 of mass \( m_1 \), starting from \( x = 0 \) with an initial speed \( v_i \), slides a distance \( d \) in the positive \( x \)-direction on a level surface with a non-uniform coefficient of kinetic friction \( \mu_k = bx^2 \), where \( b \) is a positive constant. Object 1 then continues sliding on a frictionless surface. It collides and sticks to a second object (object 2) of mass \( m_2 \) (that was initially at rest). The collision takes place over a short time interval \( \Delta t \). The downward acceleration of gravity is \( g \).

What is the magnitude of the average force that object 1 exerts on object 2 during the collision? Express your answer in terms of \( g \), \( m_1 \), \( m_2 \), \( v_i \), \( d \), \( b \), and \( \Delta t \), as needed.

Grading: Correct work integral 6 points; work energy theorem 3 points.
8 points: conservation of momentum.

\[ F_{i2} \text{ ave, } x = \frac{m_2 v_c}{\Delta t} \]

Grading: 3 points impulse, 5 points correct change in momentum.
Alternative solution

\[(F_{2,1})_{ave,x} = \frac{\Delta p_1}{\Delta t} = m_1 v_a - m_1 v_b\]

\[= m_1 \frac{m_1 v_b}{m_1 + m_2} - m_1 v_b\]

\[= -m_1 m_2 \frac{v_b}{m_1 + m_2} = -m_1 m_2 \left( \frac{v_i^2 - \frac{2}{3} b d^3 g}{m_1 + m_2} \right)\]

\[(F_{1,2})_{ave,x} = - (F_{2,1})_{ave,x}\]

\[= \frac{m_1 m_2}{m_1 + m_2} \sqrt{v_i^2 - \frac{2}{3} b d^3 g}\]
Problem 4 of 4: (25 points) Concept Questions (Parts A through E)

Part A (5 points)

Consider two carts, of masses $m$ and $2m$, at rest on an air track. If you first push one cart for a time interval $\Delta t$ and then the other for the same length of time, exerting equal force on each, the kinetic energy of the light cart is

a) one-fourth
b) one-half
c) equal to
d) twice
e) four times

the kinetic energy of the heavy car.

Enter the letter of the correct statement on the answer line below.

Answer: d)
Part B (5 points)

A skier of mass $M$ slides down a circular ramp segment of radius $R$. When the skier is at an angle of $45^\circ$ with respect to the vertical, the magnitude of the normal force exerted by the ramp on the skier is $N$. The downward acceleration of gravity is $g$.

a) The magnitude of the normal force $N$ is greater than $(\sqrt{2}/2)Mg$.

b) The magnitude of the normal force $N$ is equal to $(\sqrt{2}/2)Mg$.

c) The magnitude of the normal force $N$ is less than $(\sqrt{2}/2)Mg$.

d) The magnitude of the normal force $N$ can be greater than, equal to, or less than $(\sqrt{2}/2)Mg$ depending on the speed of the skier.

Enter the letters of the correct statement on the answer line below.

**Answer: a)**
Part C (5 points)

A person is sliding on ice at speed $v$ while holding a bag of sand that is slowly leaking from the bottom of the bag. There are no external horizontal forces acting on the person. In particular the ice is frictionless, and you may ignore air resistance. While the sand is leaking, the speed of the person

a) increases.

b) decreases.

c) stays the same.

d) first increases and then stays the same.

e) first decreases and then stays the same.

Enter the letter of the correct statement on the answer line below.

Answer: c)
Part D (5 points):

A person of mass $m$ is standing on the right end of a cart of length $s$ and the same mass $m$. The cart and the person are initially at rest. The person then walks to the left end and stops. You may assume there is zero rolling resistance between the cart and the ground. When the person is finished walking,

a) the cart has moved to the right a distance $d = s$.

b) the cart has moved to the left a distance $d = s$.

c) the cart has moved to the right a distance $d < s$.

d) the cart has moved to the left a distance $d < s$.

e) the cart has moved to the right a distance $d > s$.

f) the cart has moved to the left a distance $d > s$.

g) the center of mass of the person-cart system has stayed in the same place.

Enter the letter(s) of the correct statement(s) on the answer line below.

Answer: c) and g)
A particle is moving in a circular path of radius $R$. At $t = 0$, it is located at $\theta_0 > 0$. The angle the particle makes with the vertical is given by

$$\theta(t) = \theta_0 - Bt^2, \quad 0 < t < t_1$$

where $B > 0$. Which of the following statements best describes the tangential and the radial components of the acceleration $a_\theta$ and $a_r$ for $0 < t < t_1$?

a) $a_\theta = 0, a_r = 0$
b) $a_\theta = 0, a_r > 0$
c) $a_\theta > 0, a_r = 0$
d) $a_\theta > 0, a_r > 0$
e) $a_\theta = 0, a_r < 0$
f) $a_\theta < 0, a_r = 0$
g) $a_\theta > 0, a_r < 0$
h) $a_\theta < 0, a_r < 0$

Enter the letter of the correct statement on the answer line below.

Answer: h)