Problem 1 of 4 (25 points)

Answers without work shown will not be given any credit.

Consider a fixed wheel that consists of a uniform cylindrical disk with a radius $R$ and a mass $M_1$, and a second uniform cylindrical disk of radius $R/2$ and mass $M_2$ that is attached concentrically to the first disk, as shown in the top view (above right). The wheel is free to rotate about a frictionless pivot at its center. A rope is wound around the smaller disk, then around a small frictionless post. The other end of the rope is attached to a block of mass $M_b$ so that both disks rotate together as the block falls. The moment of inertia of a uniform disk of mass $M$ and radius $R$ about an axis passing through the center of mass and perpendicular to the plane of the disk is $(1/2)MR^2$. The downward acceleration of gravity is $g$. The block is released from rest. How long does it take the block to fall a distance $d$? Express all answers in terms of $R$, $M_1$, $M_2$, $M_b$, $g$, and $d$ as needed.
\[ I_{cm} = \frac{1}{2} m_1 R^2 + \frac{1}{2} m_2 \left( \frac{R}{2} \right)^2 = \frac{1}{2} (m_1 + m_2) R^2 \] 4 points

\[ F = m \vec{a} \]

\[ J : m_b g - T = m_b a_y \] 4 points

Constraint: \[ a_y = \frac{1}{2} R \alpha_z \] 4 points
\[ a_y = \frac{1}{2} R \alpha_z \]
\[ m_b g - T = m_b a_y \]
\[ \frac{R}{\theta} T = \frac{1}{2} (m_1 + m_2) R \alpha_z \]
\[ \Rightarrow T = 2 (m_1 + \frac{m_2}{4}) a_y \]
\[ m_b g - 2 (m_1 + m_2) a_y = m_b a_y \]

\[ \Rightarrow a_y = \frac{m_b g}{m_b + 2 (m_1 + \frac{m_2}{4})} \quad \text{1 point units} \]
\[ a_y = \frac{2 m_b g}{2 m_b + 4 m_1 + m_2} \quad \text{1 point correct answer} \]

\[ t = \sqrt{ \frac{2 d}{a_y} } = \sqrt{ \frac{2 d (4 m_1 + 2 m_b + m_2)}{2 m_b g} } \]

\[ t = \sqrt{ \frac{d (4 m_1 + 2 m_b + m_2)}{m_b g} } \quad \text{3 points} \]

\[ \{ 1 \text{ for kinematics}, \quad \{ 1 \text{ for correct dim.} \}
\[ \{ 1 \text{ for answer} \]
Problem 2 of 4 (25 points)

Answers without work shown will not be given any credit.

Particle 1 of mass \( m_1 \), initially moving in the positive \( x \)-direction (to the right in the figure below) with speed \( v_{1,i} \), collides with particle 2 of mass \( m_2 = m_1/3 \), which is initially moving in the opposite direction (to the left in the figure) with an unknown speed \( v_{2,i} \). Assume that the total external force acting on the particles is zero. **Do not assume the collision is elastic.** After the collision, particle 1 moves with speed \( v_{1,f} = v_{1,i} / 2 \) in the negative \( y \)-direction (downward in the figure). After the collision, particle 2 moves with an unknown speed \( v_{2,f} \), at an angle \( \theta_{2,f} = 45^\circ \) with respect to the positive \( x \)-direction (upward and to the right in the figure).

Note that \( \sin 45^\circ = \cos 45^\circ = \sqrt{2} / 2 \).

![Diagram](image)

**a)** Determine the initial speed \( v_{2,i} \) of particle 2 and the final speed \( v_{2,f} \) of particle 2 in terms of \( v_{1,i} \).

**b)** Is the collision elastic? Justify your answer.
\[ P_{x,i} = P_{x,f} \]
\[ m_1 v_{1,i} - m_3 v_{3,i} = m_3 v_{2,f} \frac{\sqrt{2}}{2} \quad + \text{points} \]
\[ P_{y,i} = P_{y,f} \]
\[ 0 = \frac{1}{3} m_1 v_{2,f} \frac{\sqrt{2}}{2} - m_1 v_{1,i} \frac{\sqrt{2}}{2} \quad 7 \text{points} \]
\[ \Rightarrow v_{2,f} = 3 \frac{\sqrt{2}}{2} v_{1,i} \quad 1 \text{point} \]
\[ \Rightarrow v_{1,i} - v_{2,i} = \frac{13}{3} \frac{\sqrt{2}}{2} v_{1,i} \frac{\sqrt{2}}{2} \]
\[ \Rightarrow v_{2,i} = 3 \frac{\sqrt{2}}{2} v_{1,i} \quad 1 \text{point} \]
\[ k_{c} = \frac{1}{2} m_1 v_{1,i}^2 + \frac{1}{2} \left( \frac{m_1}{3} \right) \left( \frac{3}{2} v_{1,i} \right)^2 \quad 3 \text{points} \]
\[ = \frac{1}{2} m_1 \left( \frac{7}{4} \right) v_{1,i}^2 \]
\[ k_{f} = \frac{1}{2} m_1 \left( \frac{\sqrt{2}}{2} v_{1,i} \right)^2 + \frac{1}{2} \left( \frac{m_1}{3} \right) \left( \frac{3}{2} \sqrt{2} v_{1,i} \right)^2 \quad 3 \text{points} \]
\[ = \frac{1}{2} m_1 \left( \frac{7}{4} \right) v_{1,i}^2 \]
\[ k_{c} = k_{f} \quad \Rightarrow \text{energy is constant} \]
Problem 3 of 4 (25 points)

A massless spring of spring constant $k$ is initially compressed a distance $d$ between a thin uniform rod of mass $m_r$ and length $l$, and a stop. The rod is pivoted about the point $P$. The rod is initially held in place on a frictionless table. When the rod is released it rotates until it strikes a ball of mass $m_b$ that is at rest a distance $s$ from the pivot point $P$ (see figure on left). After the collision, the ball moves with speed $v$ in the direction shown in the figure on the above right. **Assume the collision of the rod and the ball is inelastic but that they do not stick together.** The moment of inertia of a uniform rod of mass $m_r$ and length $l$ about an axis passing through the center of mass and perpendicular to the rod is $(1/12)m_r l^2$. What is the final angular speed $\omega_f$ after the collision with the ball? Express all answers in terms of $k$, $d$, $m_r$, $l$, $m_b$, $v$, and $s$ as needed.
\[ E_i = \frac{E_{\text{before collision}}}{5} \quad \text{3 points} \]

\[ \frac{1}{2} k d^2 = \frac{1}{6} I_p w_b^2 \quad \text{6 points} \]

\[ I_p = m_r \left( \frac{d}{2} \right)^2 + \frac{1}{12} m_r l^2 = \frac{1}{3} m_r l^2 \quad \text{3 points} \]

\[ \Rightarrow w_b = \sqrt{\frac{k d^2}{I_p}} = \sqrt{\frac{3 k d^2}{m_r l^2}} \]

\[ I_p \quad \delta \quad \theta \quad \text{C} \]

Angular momentum about pivot point \( \hat{P} \)

3 points

Is constant immediately before and after collision due to no exterior torques at pivot point \( \hat{P} \)

\[ \vec{L}_{p, b} = I_p \, w_b \, \hat{k} \quad \text{2 points} \]

\[ \vec{L}_{p, f} = I_p \, w_f \, \hat{k} + m_b s\, v \, \hat{k} \quad \text{4 points} \]

\[ I_p \, w_b = I_p \, w_f + m_b s\, v \quad \text{2 points} \]

\[ \Rightarrow w_f = w_b - \frac{m_b s\, v}{I_p} \]

\[ \Rightarrow w_f = \sqrt{\frac{3 k}{m_r}} \, \frac{d}{l} - \frac{3 m_b s\, v}{m_r l^2} \quad \text{2 points} \]
A uniform rigid rod is lying on a horizontal frictionless table and pivoted at the point \( P \) on the one end (shown in the figure). A point-like object is moving to the right (see figure) with speed \( v_0 \). It collides elastically with the rod at the end of the rod and rebounds backwards with speed \( v_f \). After the collision, the rod rotates clockwise about its pivot point \( P \) with angular velocity \( \vec{\omega} \). After the collision, how does the magnitude of the angular momentum of the rod about the pivot point \( P \) compare to the angular momentum of the point-like object about the pivot point \( P \)?

1) greater.

2) equal to.

3) less than.

4) Not enough information is specified to decide which of the above three statements is correct.

Enter the number of the correct statement on the answer line below.

Answer: ______1___________________
A ping-pong ball and a cannonball collide elastically. The velocity of the ping-pong ball before the collision is \( \mathbf{v}_{p,0} \hat{i} \), with \( \mathbf{v}_{p,0} > 0 \). The velocity of the cannonball before the collision is \( -\mathbf{v}_{c,0} \hat{i} \), with \( \mathbf{v}_{c,0} > 0 \). The line of motion of the ping-pong ball passes through the center of the cannonball. The velocity of the ping-pong ball after the collision is \( \mathbf{v}_{p,f} \).

You may assume that the mass of the cannonball is much greater than the mass of the ping-pong ball and ignore any effects due to gravity. After the collision, the velocity of the ping-pong ball is

1) \( \mathbf{v}_{p,f} = (\mathbf{v}_{p,0} + \mathbf{v}_{c,0}) \hat{i} \).

2) \( \mathbf{v}_{p,f} = -(\mathbf{v}_{p,0} + \mathbf{v}_{c,0}) \hat{i} \).

3) \( \mathbf{v}_{p,f} = (2\mathbf{v}_{p,0} + \mathbf{v}_{c,0}) \hat{i} \).

4) \( \mathbf{v}_{p,f} = -(2\mathbf{v}_{p,0} + \mathbf{v}_{c,0}) \hat{i} \).

5) \( \mathbf{v}_{p,f} = (\mathbf{v}_{p,0} + 2\mathbf{v}_{c,0}) \hat{i} \).

6) \( \mathbf{v}_{p,f} = -(\mathbf{v}_{p,0} + 2\mathbf{v}_{c,0}) \hat{i} \).

7) None of the above.

Enter the number of the correct statement on the answer line below.

Answer: _______ 6 ______________
Part C (5 points)

A lunar mapping satellite of mass $m_s$ is in a circular orbit around the moon, and the orbit has radius $R_0$. A repair robot of mass $m_r < m_s$ is also in a circular orbit (same radius), but in the opposite direction. The orbital speed $v_0$ of the two objects is the same. Some time later the two objects collide and stick together. Assume that the collision happens instantaneously. Consider the following three statements.

I. The sum of the angular momentum of the satellite and the robot about the center of the moon does not change as a result of the collision.

II. The mechanical energy of the satellite and robot does not change as a result of the collision.

III. The sum of the momentum of the satellite and the robot does not change as a result of the collision.

Which of the following is correct?

1) I, II, III are true.
2) I and II are true, but III is false.
3) I and III are true, but II is false.
4) II and III are true, but I is false.
5) Only I is true.
6) Only II is true.
7) Only III is true.
8) I, II, and III are false.

Enter the number of the correct statement on the answer line below.

Answer: _______3______________
Part D (5 points):

A uniform rod of length $L$ and mass $m$ is in static equilibrium, hinged to a wall at one end and suspended from the wall by a cable that is attached to the other end of the rod at an angle $\beta = 30^\circ$ with respect to the horizontal (see figure above). A block of equal mass $m$ is resting on the rod a distance $L / 4$ from the hinge point. Let $T$ denote the magnitude of the tension in the cable. Let $F_y$ denote the $y$-component of the hinge force acting at the point where the rod is hinged to the wall. Assume the cable has zero mass. Note that $\sin 30^\circ = 1 / 2$ and $\cos 30^\circ = \sqrt{3} / 2$. Which of the following statements is correct?

1) $T = \frac{3}{4}mg$ and $F_y = \frac{5}{4}mg$.
2) $T = \frac{3}{4}mg$ and $F_y = -\frac{5}{4}mg$.
3) $T = \frac{3}{2}mg$ and $F_y = \frac{5}{4}mg$.
4) $T = \frac{3}{2}mg$ and $F_y = -\frac{5}{4}mg$.
5) $T = \frac{3\sqrt{2}}{4}mg$ and $F_y = \frac{5}{4}mg$.
6) $T = \frac{5\sqrt{2}}{4}mg$ and $F_y = -\frac{5}{4}mg$.
7) None of the above.

Enter the number of the correct statement on the answer line below.

Answer: ___________3________________
Consider the potential energy \( U(x) = -ax^2 + bx^4 + U_0 \) for a particle undergoing one-dimensional motion along the \( x \)-axis. The figure above shows a graph of potential energy \( U(x) \) versus position \( x \). The mechanical energy \( E \) of the particle is indicated by a dashed line. At \( t = 0 \), the particle is somewhere between the points B and I. For later times answer the following questions.

1) Which point(s) will this particle never reach?
Answer: \( \text{A, J} \)

2) At which point(s) will the force on this particle have the largest positive \( x \)-component?
Answer: \( \text{B} \)

3) At which point(s) will the force on this particle be zero?
Answer: \( \text{C, F, H} \)

4) At which point(s) will the kinetic energy of this particle be zero?
Answer: \( \text{B, I} \)

5) At which point(s) will the kinetic energy of this particle be maximum?
Answer: \( \text{C, H} \)