A simple way to measure the speed of a bullet is with a *ballistic pendulum*, which consists of a wooden block of mass $m_1$ into which a bullet of mass $m_2$ is shot. The block is suspended from two cables, each of length $L$. The impact of the bullet causes the block and embedded bullet to swing through a maximum angle $\phi$.

**a)** Find a relation for the initial speed of the bullet as a function of $m_1$, $m_2$, $L$, $g$, and $\phi$.

**b)** Find an expression for the ratio of the lost mechanical energy (due to the collision) to the initial kinetic energy of the bullet.

**Solution:** We shall assume that the collision time between the bullet and the sand is negligibly small. We shall use two concepts to solve this problem. The collision of the bullet and the block is a completely inelastic collision. We will then use a momentum flow diagram (shown below) to analyze the collision. We use the fact that the momentum is constant to determine the speed of the block immediately after the collision in terms of the speed of the bullet and the masses of the object.

After the collision is finished, there is no non-conservative work done on our system, so the mechanical energy is constant. We can use this fact to find a relation between the height that the block and bullet reached when they came to rest and the speed of the block.
immediately after the collision. We can then put these two pieces together to find the speed of the bullet in terms of the given quantities.

a) The collision of the bullet and the block is completely inelastic. Constancy of momentum in the horizontal direction is expressed as

\[ m_2 v_0 = (m_1 + m_2)v_i \]  

where \( v_i \) is the speed of the bullet-block combination after the collision. In this situation, saying that the collision time is “negligibly small” can be taken to mean that the block moves a distance very small compared to the length of the strings. The speed immediately after the collision is then

\[ v_i = \frac{m_2}{m_1 + m_2} v_0. \]  

Once the bullet is embedded in the block, the subsequent motion has constant energy. There is an external force acting on the system, the tension in the ropes, but that force points radially inward and since the block undergoes circular motion after being struck by the bullet, the tension does no work since the tension forces in the ropes are perpendicular to the displacement,

\[ \mathbf{T} \cdot d\mathbf{r} = 0. \]  

Choose zero gravitational potential energy at the collision position. Then the initial mechanical energy is

\[ E_i = K_i = \frac{1}{2} (m_1 + m_2) v_i^2 \]  

The block reaches a final height \( h_f = L(1 - \cos \varphi) \) and the final mechanical energy is then

\[ E_f = U_f = (m_1 + m_2) g h_f = (m_1 + m_2) g L(1 - \cos \varphi). \]  

Because the mechanical energy remains constant, we have

\[ E_i = \frac{1}{2} (m_1 + m_2) v_i^2 = (m_1 + m_2) g L(1 - \cos \varphi) = E_f \]  

\[ v_i^2 = 2 g L(1 - \cos \varphi). \]  

Substituting the result of Equation (2) for the speed \( v_i \) immediately after the collision into Equation (6), we have that
\[
\left( \frac{m_2 v_0}{m_1 + m_2} \right)^2 = 2gL(1 - \cos \varphi).
\] (7)

We can now solve Equation (7) for the initial speed of the bullet,

\[
v_0 = \frac{(m_1 + m_2)}{m_2} \sqrt{2gL(1 - \cos \varphi)}.
\] (8)

b) The change in mechanical energy during the collision is given by

\[
\Delta E = K_i - K_0 = \frac{1}{2} (m_1 + m_2) v_1^2 - \frac{1}{2} m_2 v_0^2.
\] (9)

Again substitute the result from Equation (2) for the velocity \( v_1 \) immediately after the collision into Equation (9) to obtain the “lost mechanical energy,” \( -\Delta E \):

\[
-\Delta E = \frac{1}{2} m_2 v_0^2 \left(1 - \frac{m_2}{(m_1 + m_2)} \right) = K_0 \frac{m_1}{(m_1 + m_2)}.
\] (10)

The ratio of the lost mechanical energy to the initial kinetic energy is

\[
-\frac{\Delta E}{K_0} = \frac{m_1}{(m_1 + m_2)}.
\] (11)

Note that this ratio only depends on the masses and is completely independent of the initial velocity or the collision forces (if the forces are so abrupt that the collision can be taken to be “instantaneous”).