IC_W11D2-5 Table Problem Pulling a Yo-Yo Solution

A Yo-Yo of mass $m$ has an axle of radius $b$ and a spool of radius $R$. It’s moment of inertia about an axis passing through the center of the Yo-Yo can be approximated by $I_0 = (1/2)mr^2$. The Yo-Yo is placed upright on a table and the string is pulled with a horizontal force $\vec{F}$ to the right as shown in the figure.

The coefficient of static friction between the Yo-Yo and the table is $\mu_s$. What is the maximum magnitude of the pulling force, $|\vec{F}|$, for which the Yo-Yo will roll without slipping?

Solution:
The free body diagram is shown in the figure below with the force of static friction acting opposite to the applied force.

Torque equation:

$$\tau_z = Rf - bF = I\alpha_z$$  \hspace{1cm} (1)

Force equation:

$$F - f = ma_x$$  \hspace{1cm} (2)

For rolling without slipping motion along the floor,
\[ a_x = +Ra_x \]  \hspace{1cm} (3)

(Note that the positive sign is a result of our two choices of sign convention for positive rotation and linear acceleration.)

From (1) and (3),

\[ Rf - bF = \frac{1}{2}maR \]  \hspace{1cm} (4)

implies that

\[ f - \frac{b}{R}F = \frac{1}{2}ma_x \]  \hspace{1cm} (5)

Substitute Eq. (5) into Eq. (2) and solve for the pulling force,

\[ F \left(1 - \frac{b}{R}\right) = \frac{3}{2}ma_x . \]  \hspace{1cm} (6)

Therefore the \( x \)-component of the acceleration is

\[ a_x = \frac{2}{3} \frac{F}{m} \left(1 - \frac{b}{R}\right) \]  \hspace{1cm} (7)

Substitute Eq. (7) into Eq. (5) and solve for the magnitude of the friction force,

\[ f = \frac{F}{3} \left(1 + \frac{2b}{R}\right) \]  \hspace{1cm} (8)

The maximum possible value of the magnitude of the frictional force, \( f \), is given by, \( f = \mu_s mg \). So, the maximum pulling force is given by

\[ F_{\text{max}} = \frac{3\mu_s mgR}{R + 2b} . \]

Our assumption of clockwise rotation breaks down when \( F > F_{\text{max}} \) and slipping occurs.
If the string is pulled very gently, our assumption of pure clockwise rotation holds. And, the Yo-Yo rotates in the forward (clockwise) direction without any slipping.

If the string is jerked hard, our assumption of pure rotation in the clockwise direction fails, and slippage occurs. The Yo-Yo rotates in the anti-clockwise direction but still moves forward (by slipping).