Consider a simple rigid body consisting of two particles of mass $m$ separated by a rod of length $2l$ and negligible mass. The midpoint of the rod is attached to a vertical axis that rotates with angular velocity $\vec{\omega} = \omega \hat{k}$ about the $z$-axis. The rod is skewed from the vertical at an angle $\phi$. Set time $t = 0$ when the rod is in the position shown in figure below left. At $t = \pi / \omega$ the rod has rotated to the position shown in the figure below right.

### a) Find the direction and magnitude of the angular momentum about the center of mass at $t = 0$.

### b) Find the direction and magnitude of the torque about the center of mass at time $t = 0$.

**Solution:**

**a)** We use $\vec{L}_{cm} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$ for the angular momentum about the center of mass for each particle:

$$\vec{L}_{cm} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2$$

Since each particle travels at an angular speed $\omega$ in a circular orbit of radius $\ell \cos \phi$, the speed of each particle is given by $v = \omega \ell \cos \phi$. We choose a coordinate system shown in the figure below.
For particle 1: \( \mathbf{r}_1 = -l \cos \phi \mathbf{i} + l \sin \phi \mathbf{k} \) and \( \mathbf{v}_1 = -l \cos \phi \omega \mathbf{j} \). Thus

\[
\mathbf{L}_{\text{cm},1} = \mathbf{r}_1 \times m \mathbf{v}_1 = (\mathbf{r}_1 \times m \mathbf{v}_1) = (-l \cos \phi \mathbf{i} + l \sin \phi \mathbf{k}) \times (-ml \cos \phi \omega \mathbf{j}).
\]

After calculating the cross products, we have that the angular momentum about the center of mass for particle 1 is

\[
\mathbf{L}_{\text{cm},1} = m l^2 \omega \cos \phi (\cos \phi \mathbf{k} + \sin \phi \mathbf{i}).
\]

Note that for particle 2, \( \mathbf{r}_2 = -\mathbf{r}_1 \) and \( \mathbf{v}_2 = -\mathbf{v}_1 \), so

\[
\mathbf{L}_{\text{cm},2} = \mathbf{r}_2 \times m \mathbf{v}_2 = \mathbf{r}_1 \times m \mathbf{v}_1 = \mathbf{L}_{\text{cm},1}.
\]

Thus the angular momentum about the center of mass at time \( t = 0 \) is given by

\[
\mathbf{L}_{\text{cm}}(0) = 2 ml^2 \omega \cos \phi (\cos \phi \mathbf{k} + \sin \phi \mathbf{i}).
\]

The magnitude of the angular momentum about the center of mass is given by

\[
L_{\text{cm}} = 2 ml^2 \omega \cos \phi (\cos^2 \phi + \sin^2 \phi) = 2 ml^2 \omega \cos \phi.
\]

b) At \( t = 0 \), the rod is rotating in the \( x - y \) plane. The figure below shows the orientation of the rod as seen from above.

The \( z \)-component of the angular momentum about the center of mass is constant and the \( x \)-component of the angular momentum about the center of mass is changing in time as the rod rotates and is given by

\[
\mathbf{L}_{\text{cm},x}(0) = 2 ml^2 \omega \cos \phi \sin \phi \mathbf{i}
\]

The time derivative of the angular momentum about the center of mass is perpendicular to the angular momentum, points in the positive \( y \)-direction, and has a magnitude that is equal to \( |\mathbf{L}_{\text{cm},x}(0)| \omega \). Therefore
\[
\frac{d\mathbf{L}_{cm}}{dt}(0) = \left| \mathbf{L}_{cm,x}(0) \right| \omega \hat{j} = 2ml^2 \omega^2 \cos \phi \sin \phi \hat{j}.
\]

The torque about the center of mass is given by

\[
\mathbf{\tau}_{cm} = \frac{d\mathbf{L}_{cm}}{dt}
\]

Therefore at \( t = 0 \), we that

\[
\mathbf{\tau}_{cm} = 2ml^2 \omega^2 \cos \phi \sin \phi \hat{j}.
\]