Small Oscillations Solutions

The force of interaction between a particle of mass \( m_1 \) and a second particle of mass \( m_2 \) separated by a distance \( r \) is given by an attractive gravitational force and a repulsive force that is proportional to \( r^{-3} \),

\[
F(r) = -\frac{Gm_1 m_2}{r^2} + C \frac{1}{r^3},
\]

where \( C \) is a positive constant. Suppose \( m_2 >> m_1 \), so you can assume that the heavier particle is at rest.

a) Choose your zero point for potential energy at infinity. If the lighter object starts off an infinite distance from the heavier object and then moves until the two objects are a distance \( r \) apart, what is the potential energy difference \( U(r) - U(\infty) = -\int_{\infty}^{r} \mathbf{F} \cdot d\mathbf{r} \) ?

b) What is the distance \( r_0 \) between the two masses when they are in stable equilibrium? What is the value of the potential energy \( U(r_0) \) at stable equilibrium?

c) Calculate the angular frequency of small oscillations about the stable equilibrium position.

d) For a fixed value of energy, \( U(r_0) < E < 0 \), find the closest distance and furthest distance between the particles.
a) \[ U(r) - U(\infty) = -\int_{\infty}^{r} f(r') dr' = -\int_{\infty}^{r} \frac{-6m_1m_2 + c}{r^2} \, dr' = \frac{-6m_1m_2}{r} \bigg|_{\infty}^{r} + \frac{c}{2r^2} \bigg|_{\infty}^{r} = \frac{-6m_1m_2}{r} + \frac{c}{2r^2} \]

b) Stable equilibrium occurs when the force between the particles is zero.

\[ 0 = \frac{-6m_1m_2}{r_0^2} + \frac{c}{r_0^3} \Rightarrow r_0 = \frac{c}{6m_1m_2} \]

\[ U(r_0) = \frac{-6m_1m_2}{r_0} + \frac{c}{2r_0^2} = \frac{-6m_1m_2}{r_0} + \frac{c}{2} \left( \frac{c}{6m_1m_2} \right)^2 \]

\[ = -\frac{1}{2} \left( \frac{6m_1m_2}{c} \right)^2 \]
C) \[ V(r) = V(r_0) + \frac{1}{2} \frac{d^2 V}{dr^2} \bigg|_{r=r_0} (r-r_0)^2 + \ldots. \]

Angular frequency of small oscillations is given by:

\[ \omega_0 = \sqrt{\frac{d^2 V}{dr^2} \bigg|_{r=r_0}} \]

\[ \frac{d^2 V}{dr^2} \bigg|_{r=r_0} = \left(-\frac{2Gm_1m_2}{r^3} + \frac{3C}{r^4} \right) \bigg|_{r=r_0} = \frac{-2Gm_1m_2 + 3C}{r_0^3} \]

\[ = \frac{-2Gm_1m_2 + 3C}{(c/6Gm_1m_2)^3} \left(\frac{cGm_1}{6m_1m_2} \right)^4 \]

\[ \omega_0 = \sqrt{\left(\frac{cGm_1^3m_2^4}{c^3} \right)} \]

\[ \frac{c}{2r^2} \]

\[ E = \]

\[ E = V(r_1) = V(r_2) \Rightarrow E = -\frac{Gm_1m_2 + C}{r} \quad \text{for } r = r_1 \]

\[ \Rightarrow \frac{-r + Gm_1m_2 - C}{E} = 0 \Rightarrow \]

\[ \frac{r}{E} = \frac{Gm_1m_2}{2E} \]

\[ r = \frac{-Gm_1m_2 \pm \left(\frac{(Gm_1m_2)^2 + 4E}{E} \right)^{1/2}}{2} \]

Two solutions for \( r \).