Problem 1:

a) Explain why the total force on a system of particles is only due to the sum of external forces.

Answer: All internal forces are part of interaction pairs and by Newton’s Third Law the sum of the forces of each interaction pair is zero hence all internal forces all cancel in pairs. Therefore only the external forces sum to give the force on a system of particles.

b) Explain under what conditions does the center of mass of a system of particles move with constant velocity?

Answer: Using Newton’s Second and Third Laws, the external force on a system of a particles is equal to the product of mass of the system with the acceleration of the center of mass of the he system. If the external force is zero, the acceleration of the center of mass is zero and hence the center of mass of a system of particles move with constant velocity.

c) Explain what is meant by the expression “the thrust on a rocket” in terms of the relative speed of the ejected fuel and the rate that fuel is burned. How is this thrust related to the conservation of momentum when there are no external forces acting on the system consisting of rocket and fuel?

Answer: The magnitude of the thrust on a rocket is the product of the time rate of change that mass of the fuel that is ejected from the rocket with the relative speed of that fuel with respect to the rocket. The direction of the thrust is in the direction of the acceleration of the rocket. The thrust term is actually the time rate of change in the momentum of the burned fuel. The direction of this change in momentum of the fuel is opposite the direction of the acceleration of the rocket. If there are no external forces acting on the rocket, the change in the momentum of the rocket is equal in magnitude to the change in momentum of the fuel but opposite in direction as the rocket recoils forward.

d) The same force $\vec{F}$ is applied separately to each of the three objects shown on the right with masses satisfying $m_1 > m_2 > m_3$. Which of the following statements is true? (Note: more than one statement may be true.)
1. In all cases the center of mass of the system has the same acceleration.

2. In all cases the magnitudes of the acceleration of the center of mass of the system are different.

3. The magnitude of the acceleration of the center of mass of the system is largest in figure (1).

4. The magnitude of the acceleration of the center of mass of the system is largest in figure (2).

5. The magnitude of the acceleration of the center of mass of the system is largest in figure (3).

Explain your reasoning.

Answer: 1. In each case the external force on the system is the same, so the center of mass has the same acceleration.
Problem 2: Jumping off the Ground

A person, of mass $m$, jumps off the ground. Suppose the person pushes off the ground with a constant force of magnitude $F$ for a time interval $\Delta t$. What was the magnitude of the maximum displacement of the center of mass of the person?

Solution: We divide the jump into two parts, pushing off the ground, and then in the air. For the first part, we begin by computing how much the center of mass displaces during the jumping. Let’s set $t = 0$ at the beginning of the jump. During the time interval $[0 < t < \Delta t]$, the person is pushing against the ground. We can first compute the acceleration of the center of mass using

$$\vec{F}_{ext} = \vec{p}_{cm}(t) - \vec{p}_{cm}(t = 0) = m\vec{v}_{cm}(t).$$  \hspace{1cm} (1)$$

Let’s choose positive $y$-axis pointing up then the external force is given by

$$\vec{F}_{ext} = \vec{F}_{ground, person} + \vec{F}_{gravity, person} = (F - mg)\hat{j}. \hspace{1cm} (2)$$

Therefore substituting Eq. (2) into Eq. (1) and taking the y-components yields

$$(F - mg)t = mv_{cm,y}(t). \hspace{1cm} (3)$$

Thus the $y$-component of the velocity of the center of mass is given by

$$v_{cm,y,1}(t) = \left(\frac{F}{m} - g\right)t; \quad 0 \leq t \leq \Delta t. \hspace{1cm} (4)$$

We can integrate this to find the displacement of the center of mass

$$\Delta y_{cm,1} = y_{cm,1}(\Delta t) - y_{cm,1}(0) = \int_{t' = 0}^{t' = \Delta t} v_{cm,y,1}(t')dt'$$

$$\Delta y_{cm,1} = \int_{t' = 0}^{t' = \Delta t} \left(\frac{F}{m} - g\right)t' dt' = \frac{1}{2}\left(\frac{F}{m} - g\right)(\Delta t)^2. \hspace{1cm} (5)$$

The $y$-component of the velocity of the center of mass when the jump is finished at $t = \Delta t$ is then

$$v_{cm,y,1}(\Delta t) = \left(\frac{F}{m} - g\right)\Delta t. \hspace{1cm} (6)$$
For the second part, once the person’s feet leave the ground, the pushing force \( F = 0 \), so the acceleration is now

\[
a_{cm,y,2} = -g .
\] (7)

We now reset our clock to zero. Integrating Eq. (7) yields the change in the \( y \)-component of the velocity of the center of mass

\[
v_{cm,y,2}(t) - v_{cm,y,2}(0) = \int_{t' = 0}^{t' = t} -g \, dt' = -gt .
\] (8)

We note that \( v_{cm,y,1}(\Delta t) = v_{cm,y,2}(0) \) and so we can solve Eq. (8) for the \( y \)-component of the velocity of the center of mass.

\[
v_{cm,y,2}(t) = v_{cm,y,1}(\Delta t) = -gt .
\] (9)

The person reaches the highest point of the jump when

\[
0 = v_{cm,y,2}(t_{up}) = v_{cm,y,1}(\Delta t) - gt_{up}
\] (10)

Therefore

\[
t_{up} = v_{cm,y,1}(\Delta t) / g
\] (11)

The additional displacement is given by integrating Eq. (8)

\[
\Delta y_{cm,2} = \int_{t' = 0}^{t' = \Delta t} v_{cm,y,2}(t') \, dt' = \int_{t' = 0}^{t' = \Delta t} (v_{cm,y,1}(\Delta t) - gt') \, dt'
\] (12)

\[
= v_{cm,y,1}(\Delta t) t_{up} - (1/2)gt_{up}^2
\]

Substituting Eq. (11) and Eq. (6) into Eq. (12) yields

\[
\Delta y_{cm,2} = (1/2)v_{cm,y,1}(\Delta t)^2 / g = \frac{1}{2g} \left( \frac{F}{m} - g \right)^2 \Delta t^2
\] (13)

So the maximum displacement of the center of mass of the person through both parts is determined by adding Eqs. (5) and (13),

\[
\Delta y_{cm,max} = \Delta y_{cm,1} + \Delta y_{cm,2} = \frac{1}{2} \left( \frac{F}{m} - g \right) (\Delta t)^2 + \frac{1}{2g} \left( \frac{F}{m} - g \right)^2 \Delta t^2
\] (14)
A little algebra then yields

\[ \Delta y_{cm,\text{max}} = \frac{F(\Delta t)^2}{2mg} \left( \frac{F}{m - g} \right). \]

(15)

**Problem 3: Center of Mass of a Rod**

A thin rod has length \( L \) and total mass \( M \).

a) Suppose the rod is uniform. Find the position of the center of mass with respect to the left end of the rod.

b) Now suppose the rod is not uniform but has a linear mass density that varies with the distance \( x \) from the left end according to

\[ \lambda = \frac{\lambda_0}{L^2} x^2 \]

(1)

where \( \lambda_0 \) is a constant and has SI units [kg \cdot m\(^{-1}\)]. Find \( \lambda_0 \) and the position of the center of mass with respect to the left end of the rod.

**Solution:** (a) Choose a coordinate system with the rod aligned along the x-axis and origin located at the left end of the rod. The center of mass of the rod can be found using the definition

\[ \mathbf{R}_{\text{cm}} = \frac{1}{M} \int_{\text{body}} \mathbf{r} \, dm = \frac{\int_{\text{body}} \mathbf{r} \, dm}{\int_{\text{body}} dm} \]

(2)

In this expression \( dm \) is an infinitesimal mass element and \( \mathbf{r} \) is the vector from the origin to the mass element \( dm \).

Choose an infinitesimal mass element \( dm \) located a distance \( x' \) from the origin. In this problem \( x' \) will be the integration variable. Let the length of the mass element be \( dx' \). Then
\[ dm = \frac{M}{L} \, dx' \quad (3) \]

The vector \( \vec{r} = x' \hat{i} \). The center of mass is found by integration

\[ \vec{R}_{cm} = \frac{1}{M} \int_{\text{body}} \vec{r} \, dm = \frac{1}{L} \int_{x=0}^{L} x' \, dx' \hat{i} = \frac{1}{2L} \left[ x'^3 \right]_{x'=0}^{x'=L} \hat{i} = \frac{1}{2L} (L^2 - 0) \hat{i} = \frac{L}{2} \hat{i} \quad (4) \]

(b) For a non-uniform rod, the mass element is found using Eq. (1)

\[ dm = \lambda(x') \, dx' = \frac{\lambda_0}{L} x'^2 \, dx' \quad (5) \]

The vector \( \vec{r} = x' \hat{i} \). The total mass is found by integrating the mass element over the length of the rod

\[ M = \int_{\text{body}} dm = \int_{x=0}^{x=L} \frac{\lambda_0}{L} x'^2 \, dx' = \frac{\lambda_0}{3L} \left[ x'^3 \right]_{x'=0}^{x'=L} = \frac{\lambda_0}{3L} (L^3 - 0) = \frac{\lambda_0}{3} L \quad (6) \]

Therefore

\[ \lambda_0 = \frac{3M}{L} \quad (7) \]

The center of mass is again found by integration

\[ \vec{R}_{cm} = \frac{1}{M} \int_{\text{body}} \vec{r} \, dm = \frac{3}{\lambda_0 L} \int_{x'=0}^{x'} \lambda(x') x' \, dx' \hat{i} = \frac{3}{L} \int_{x'=0}^{x'} x'^2 \, dx' \hat{i} \]

\[ \vec{R}_{cm} = \frac{3}{4L^2} \left[ x'^3 \right]_{x'=0}^{x'=L} \hat{i} = \frac{3}{4L^2} (L^3 - 0) \hat{i} = \frac{3}{4} L \hat{i} \quad (8) \]
Problem 4 Walking on a Cart

A person of mass \( m_p \) is standing at the right end of a cart of mass \( m_c \) and length \( s \). The cart and the person are initially at rest. The person then walks to the left end and stops. You may assume that there is zero rolling resistance between the cart and the ground.

a) What is the speed of the cart when the person has finished walking?

b) When the person is finished walking in what direction and how far has the cart moved?

c) If the person takes a time interval \( \Delta t_i \) to go from rest to a speed \( u \) relative to the cart, what is the average force the person exerts on the cart during this time interval? Express your answer only in terms of \( m_p \), \( m_c \), \( \Delta t_i \), and \( u \) as needed.

Solution:

a) There are no external horizontal forces acting on the system consisting of the person and the cart. Because the momentum of the cart and person is is zero before the person started walking, it also must be zero after the person has stopped walking. Therefore the final speed of the cart must be zero.

b) Because the center of mass was initially at rest and there are no external forces in the horizontal direction, the center of mass does not accelerate and hence remains at rest and in the same place. So we will calculate the position of the center of mass before the person started walking and after the person finished walking, and set them equal to find out how far the cart has moved to the right. Chose a coordinate system with the origin at the left end of the cart before the person has started to walk.
Then the center of mass is located at

\[ x_{cm} = \frac{m_c x_{cm,\text{cart}} + m_p x_{p,o}}{m_c + m_p} \]

We assume the cart is uniform so \( x_{cm,\text{cart}} = s / 2 \) and the person is initially standing at the end of the cart so \( x_{p,o} = s \). Therefore

\[ x_{cm} = \frac{m_c s / 2 + m_p s}{m_c + m_p}. \] (1)

We the person reaches the left end of the cart and stops, the cart has moved a distance \( x_{p,f} \). The center of mass of the cart is now located at \( x_{cm,\text{cart}} = x_{p,f} + s / 2 \). Therefore

\[ x_{cm} = \frac{m_c (x_{p,f} + s / 2) + m_p x_{p,f}}{m_c + m_p}. \] (2)

Now we equate Eqs. (1) and (2)

\[ \frac{m_c s / 2 + m_p s}{m_c + m_p} = \frac{m_c (x_{p,f} + s / 2) + m_p x_{p,f}}{m_c + m_p} \]

and find that the final position of the person and also the distance that cart has moved is
\[ x_{p,f} = \frac{m_p \Delta x}{m_c + m_p}. \]  

\[ \Delta \ell \]

c) If the person takes a time interval \( \Delta t \) to go from rest to a speed \( u \) relative to the cart, what is the average force the person exerts on the cart during this time interval? Express your answer only in terms of \( m_p \), \( m_c \), \( \Delta t \), and \( u \) as needed.

The average force that the person exerts on the cart has equal magnitude and opposite direction of the average force that the cart exerts on the person. The impulse that the car exerts on the person is equal to the change in momentum of the person. Therefore

\[ I_{x,ave} = (F_{x,ave})_{person,cart} \Delta t = \Delta p_{x, person}. \]  

\[ \Delta \ell \]

When the person is walking to the left with speed \( u \) relative to the cart, and the cart is moving to the right with speed \( v_{c,g} \) relative to the ground. Then the person is moving to the left with speed \( v_{p,g} = u - v_{c,g} \) relative to ground. The momentum of the person and the cart is zero relative to the ground and so

\[ 0 = \Delta p_{x, person} = -m_p v_{p,g} + m_c v_{c,g} = -m_p (u - v_{c,g}) + m_c v_{c,g}. \]

Therefore we can solve for the speed of the cart relative to the ground

\[ v_{c,g} = \frac{m_p}{m_c + m_p} u. \]

Hence the speed of the person relative to the ground is

\[ v_{p,g} = u - v_{c,g} = u - \frac{m_p}{m_c + m_p} u = \frac{m_c}{m_c + m_p} u. \]

The change in momentum of the person is

\[ \Delta p_{x, person} = -m_p v_{p,g} = -\frac{m_p m_c}{m_p + m_c} u. \]  

\[ \Delta \ell \]

Substitute Eq. (5) into Eq. (4) and solve for the average force of the person by the cart

\[ (F_{x,ave})_{person,cart} = -\frac{m_p m_c}{m_p + m_c} \frac{u}{\Delta t}. \]  

\[ \Delta \ell \]
By Newton’s Third Law, the force on the cart by the person is then

\[(F_{x,ave})_{cart, person} = -(F_{x,ave})_{person, cart} = \frac{m_p m_c}{m_p + m_c} \frac{u}{\Delta t_i}. \quad (7)\]
Problem 5 Stopping a Bullet

A bullet of mass $m_1$ traveling horizontally with speed $u$ is stopped in a block of mass $m_2$ that is originally at rest. Assume that the collision is nearly instantaneous. The block then slides horizontally a distance $d$ on a surface with kinetic friction coefficient $\mu_k$, and then falls off the surface at a height $h$ as shown. Neglect air resistance. Gravity with constant gravitational acceleration $g$ is acting on the system. Assume that all distances are large compared to the size of the block, and ignore small corrections that arise from the finite size of the block.

a) Determine a condition on the initial bullet speed $u$ so that the block falls off the surface.

b) Assume that the initial speed of the bullet $u$ is large enough for the block to fall off the surface (as shown). How far $x_f$ from the bottom edge of the cliff does the block hit the ground?

Solution: We first analyze the collision in order to determine the speed $v_a$ of the bullet and block after the collision. Then we will use Newton’s Second Law to analyze the sliding motion along the surface to determine the speed of the bullet and block when it just leaves the surface. We can finally use projectile motion kinematics to determine where the block and bullet hit the ground.

Let’s choose positive $x$-direction to the right (direction of motion of bullet). During the collision,
\[ F_{ext,x} \Delta t_{coll} = \Delta p_{sys,x}. \]  

(1)

Because we are assuming that the collision is nearly instantaneous, we have that

\[ 0 = (m_1 + m_2)v_a - m_1 u. \]  

(2)

Therefore the speed of the bullet and block after the collision is

\[ v_a = \frac{m_1}{m_1 + m_2} u. \]  

(3)

While the bullet and block are sliding on the surface, the force diagram is shown in the figure below.

There is a kinetic friction force in the horizontal direction so Newton’s Second Law in the \( x \)-direction is

\[ -u_k (m_1 + m_2) g = (m_1 + m_2)a_x. \]  

(4)

The acceleration is therefore

\[ a_x = -u_k g. \]  

(5)

Let’s set our clock \( t = 0 \) immediately after the collision, and choose our origin at the collision point as shown in the figure below.

Then integrating Eq. (5) yields
\[
v_x(t) - v_x(0) = \int_{t'=0}^{t'=t} a_x \, dt' = \int_{t'=0}^{t'=t} -u_k \, g \, dt' = -u_k \, gt.
\] (6)

Note that \(v_x(0) = v_a\) and so we can solve Eq. (6) for the \(x\)-component of the velocity.

\[
v_x(t) = v_a - u_k \, gt.
\] (7)

We can integrate \(v_x(t)\) to find the displacement

\[
x(t) - x(0) = \int_{t'=0}^{t'=t} v_x(t') \, dt' = \int_{t'=0}^{t'=t} (v_a - u_k \, gt') \, dt' = v_a \, t - \frac{1}{2} u_k \, g t^2.
\] (8)

Let \(t_1\) denote the time that the bullet and block reach the edge of the level surface, and using \(x(t_1) = d\) and \(x(0) = 0\), Eq. (8) becomes

\[
d = v_a \, t_1 - \frac{1}{2} u_k \, g t_1^2.
\] (9)

Setting \(t = t_1\) in Eq. (7) and solve for \(t_1\):

\[
t_1 = \frac{v_a - v_x(t_1)}{u_k \, g}.
\] (10)

Substituting Eq. (10) into Eq. (9) yields

\[
d = v_a \left( \frac{v_a - v_x(t_1)}{u_k \, g} \right) - \frac{1}{2} u_k \, g \left( \frac{v_a - v_x(t_1)}{u_k \, g} \right)^2 = \frac{1}{2} \frac{v_a^2}{u_k \, g} - \frac{1}{2} \frac{v_x(t_1)^2}{u_k \, g}.
\] (11)

We can now solve Eq. (11) for the \(x\)-component of the velocity of the bullet and the block at time \(t_1\)

\[
v_x(t_1) = \sqrt{v_a^2 - 2d \, u_k \, g}.
\] (12)

Substitute Eq. (3) into Eq. (12) yielding

\[
v_x(t_1) = \sqrt{\left( \frac{m_1}{m_1 + m_2} \right)^2 - 2d \, u_k \, g}.
\] (13)
We now apply our kinematic equations for parabolic motion when the bullet and block leave the edge of the surface. Reset our origin at the base of the vertical wall and set $t = 0$ when the bullet and block just leave the edge.

Then when the bullet and block hit the ground at $t = t_f$

$$0 = y(t_f) = h - \frac{1}{2}gt_f^2$$

(14)

$$x(t_f) = v_x(t_1)t_f$$

(15)

We can solve Eq. (14) for $t_f$

$$t_f = \sqrt{\frac{2h}{g}}$$

(16)

Substitute Eq. (16) into Eq. (15) in order to find out where the bullet and block hit the ground

$$x(t_f) = v_x(t_f)t_f = v_x(t_1)\sqrt{\frac{2h}{g}}$$

(17)

Now substitute Eq. (13) into Eq. (17) yielding our result that the bullet and block hit the ground at

$$x(t_f) = \sqrt{\frac{2h}{g} \left( \frac{m_1}{m_1 + m_2} u \right)^2 - 4hd\mu_k}$$

(18)
Problem 6

A rocket accelerates upward in a uniform gravitational field pointing downwards with constant $g = 9.8 \text{ m/s}^2$. The rocket has a dry mass (empty of fuel) $m_{r,0} = 3.6 \times 10^7 \text{ kg}$, and initially carries fuel with mass $m_{f,0} = 4.2 \times 10^7 \text{ kg}$. The fuel is ejected at a speed $u = 2.2 \times 10^3 \text{ m/s}$ relative to the rocket. The fuel burn time is $\Delta t_{\text{burn}} = 160 \text{ s}$. What is the final speed of the rocket after all the fuel has burned?

Solution

Choose the positive $x$-direction for the direction of motion of the rocket. The rocket burns fuel that is then ejected backward with velocity $\mathbf{u} = -u \hat{i}$ relative to the rocket, where $u > 0$ is the relative speed of the ejected fuel. This exhaust velocity is independent of the velocity of the rocket. The rocket must exert a force to accelerate the ejected fuel backwards and therefore by Newton’s Third law the fuel exerts a force that is equal in magnitude but opposite in direction resulting in propelling the rocket forward. The rocket velocity is a function of time, $v_{r}(t) = v_{r,x}(t) \hat{i}$, and the $x$-component increases at a rate $dv_{r,x}/dt$. Because fuel is leaving the rocket, the mass of the rocket is also a function of time, $m_{r}(t)$, and is decreasing at a rate $dm_{r}/dt$.

Let $t = t_i$ denote the instant the rocket begins to burn fuel and let $t = t_f$ denote the instant the rocket has finished burning fuel. At some arbitrary time $t$ during this process, the rocket has velocity $\mathbf{v}_{r}(t) = v_{r,x}(t) \hat{i}$ where and the mass of the rocket is $m_{r}(t) = m_r$. During the time interval $[t,t+\Delta t]$, where $\Delta t$ is taken to be small interval, a small amount of fuel of mass $\Delta m_f$ is ejected backwards with speed $u$ relative to the rocket. The fuel was initially traveling at the speed of the rocket and so undergoes a change in momentum. Also the rocket recoils forward also undergoing a change in momentum. In order to keep track of all momentum changes, we define our system to be the rocket and the small amount of fuel that is ejected during the interval $\Delta t$. At time $t$, the fuel has not yet been ejected so it is still inside the rocket. The figure below represents a momentum diagram at time for our system relative to a fixed inertial reference frame.

![Momentum Diagram](image)

The $x$-component of the momentum of the system at time $t$ is therefore
\[ p_{\text{sys},x}(t) = (m_r(t) + \Delta m_f) v_{r,x}(t). \] (1)

During the interval \([t, t + \Delta t]\) the fuel is ejected backwards relative to the rocket with speed \(u\). The rocket recoils forward with an increased \(x\)-component of the velocity \(v_{r,x}(t + \Delta t) = v_{r,x}(t) + \Delta v_{r,x}\), where \(\Delta v_{r,x}\) represents the increase the rocket’s \(x\)-component of the velocity. As usual let’s assume that all the fuel element with mass \(\Delta m_f\) has completely left the rocket at the end of the time interval, so the \(x\)-component of the velocity of the rocket is \(v_{f,x} = v_{r,x} + \Delta v_{r,x} - u\). The momentum diagram of the system at time \(t + \Delta t\) is shown in the figure below.

The \(x\)-component of the momentum of the system at time \(t + \Delta t\) is therefore

\[ p_{\text{sys},x}(t + \Delta t) = m_r(t) (v_{r,x}(t) + \Delta v_{r,x}) + \Delta m_f (v_{r,x}(t) + \Delta v_{r,x} - u). \] (2)

In the figure below we show the diagram depicting the change in the \(x\)-component of the momentum of the rocket and fuel.

\[
\Delta p_{f,x} = \Delta m_f (\Delta v_{r} - u) \\
\Delta p_{r,x} = m_r(t) \Delta v_{r,x}
\]

Therefore the change in the \(x\)-component of the momentum of the system is given by

\[
\Delta p_{\text{sys},x} = \Delta p_{r,x} + \Delta p_{f,x} = m_r(t) \Delta v_{r,x} + \Delta m_f (\Delta v_{r,x} - u). \] (3)

We again note that \(\Delta p_{f,x} = \Delta m_f (\Delta v_{r,x} - u) = -\Delta m_f u\), and we show the modified diagram for the change in the \(x\)-component of the momentum of the system in the figure below.
We can now apply Newton’s Second Law in the form of the momentum, for the system consisting of the rocket and exhaust fuel:

\[
F_{\text{ext},x} = \lim_{\Delta t \to 0} \frac{p_{\text{sys},x}(t + \Delta t) - p_{\text{sys},x}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p_{\text{sys},x}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p_{r,x}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta p_{f,x}}{\Delta t}.
\] (4)

From our diagram depicting the change in the \( x \)-component of the momentum of the system, we have that

\[
F_{\text{ext},x} = \lim_{\Delta t \to 0} \frac{m_r(t)\Delta v_{r,x}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_f (\Delta v_{r,x} - u)}{\Delta t}.
\] (5)

We note that \( \Delta m_f \Delta v_{r,x} \) is a second order differential, therefore

\[
\lim_{\Delta t \to 0} \frac{\Delta m_f \Delta v_{r,x}}{\Delta t} = 0.
\] (6)

We also note that

\[
\frac{dv_{r,x}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v_{r,x}}{\Delta t},
\] (7)

and

\[
\frac{dm_f}{dt} = \lim_{\Delta t \to 0} \frac{\Delta m_f}{\Delta t}.
\] (8)

Therefore Eq. (5) becomes

\[
F_{\text{ext},x} = m_r(t) \frac{dv_{r,x}}{dt} - \frac{dm_f}{dt} u.
\] (9)

The rate of decrease of the mass of the rocket, \( \frac{dm_r}{dt} \), is equal to the negative of the rate of increase of the exhaust fuel

\[
\frac{dm_r}{dt} = -\frac{dm_f}{dt}.
\] (10)

Therefore substituting Eq. (10) into Eq. (9), we find that the differential equation describing the motion of the rocket and exhaust fuel is given by
\[ F_{\text{ext},x} = \frac{dm_r}{dt} u = m_r(t) \frac{dv_{r,x}}{dt}. \] (11)

Eq. (11) is called the *rocket equation*. For a rocket launch at the surface of the earth

\[ F_{\text{ext},x}(t) = -m_r(t) g. \] (12)

Substitute Eq. (12) into Eq. (11) and multiply each side by \( dt \):

\[-m_r(t) g dt = m_r(t) dv_r + dm_r u\] (13)

Now divide through by \( m_r(t) \) and after rearranging terms we have

\[ dv_r = -g dt - \frac{dm_r}{m_r(t)} u \] (14)

We now integrate both sides

\[ \int_{v_i}^{v_f} dv_r = \int_{m_{r,i}}^{m_{r,f}} \frac{dm_r}{m_r(t)} u \int_{0}^{t_f} g dt. \] (15)

where \( m_{r,i} \) is the initial mass of the rocket and the fuel. Integration yields

\[ v_{r,f}(t_f) = -u \ln \left( \frac{m_{r,f}}{m_{r,i}} \right) - gt_f. \] (16)

After all the fuel is burned at \( t = t_f \), the mass of the rocket is equal to the dry mass \( m_{r,f} = m_{r,d} \) and so

\[ v_{r,f}(t_f) = u \ln R - gt_f. \] (17)

The initial mass of the rocket included the fuel is

\[ m_{r,i} = m_{r,0} + m_{f,0} = 3.6 \times 10^7 \text{ kg} + 4.2 \times 10^7 \text{ kg} = 7.8 \times 10^7 \text{ kg} \] (18)

The ratio of the initial mass of the rocket (including the mass of the fuel) to the final dry mass of the rocket (empty of fuel) is
\[
R = \frac{m_{r,i}}{m_{r,d}} = \frac{7.8 \times 10^7 \text{kg}}{3.6 \times 10^7 \text{kg}} = 2.2
\]  

The final speed of the rocket is then

\[
v_{r,f} = u \ln R - gt_f = (2.2 \times 10^3 \text{m} \cdot \text{s}^{-1}) \ln(2.2) - (9.8 \text{ m} \cdot \text{s}^{-2})(160 \text{ s}) = 1.33 \times 10^2 \text{ m} \cdot \text{s}^{-1} .
\]
Problem 7 Filling a Freight Car

A freight car of mass $m_{c,0}$, open at the top, is coasting along a level track with negligible friction at a speed $v_{c,0}$, when it begins to rain hard. The raindrops fall vertically with respect to the ground.

a) In a time interval $[t, t+\Delta t]$, an amount of water $\Delta m_w$ enters the freight car. Choose a system. Is a component of the momentum of your system constant? Write down a differential equation that results from the analysis of the momentum changes inside your system, in terms of the mass $m_c$ of the freight car and rain at time $t$ the horizontal component $v_c$ of the velocity of the freight car at time $t$, the infinitesimal change $dm_c$ in the mass of the freight car due to the added rain, and the infinitesimal change $dv_c$ in the horizontal component of the velocity of the freight car.

b) What is the speed of the freight car when it has collected a mass $m_w$ of rain? You may solve this part by integrating your differential equation or by using an alternate method. If you use an alternate method, please state the physical principles that are involved.

Solution.

Choose positive $x$-direction to the right in the figure below. Define the system at time $t$ to be the car with whatever rain is in it and the rain that falls into it during the time interval $[t, t+\Delta t]$. Denote the mass of the car and rain at time $t$ by $m_c(t)$ and let $\Delta m_w$ denote the rain that falls into the car during the time interval $[t, t+\Delta t]$. The rain has no $x$-component of velocity. At time $t$, the car is moving with $x$-component of the velocity $v_c(t)$. At time $t+\Delta t$ the car is moving with $x$-component of the velocity $v_c(t)+\Delta v_c$. The momentum diagrams for time $t$ and $t+\Delta t$ are shown below.
The diagram representing the change in the change in the $x$-component of the momentum is shown below.

There are no external forces in the $x$-direction acting on the system, $F_{\text{ext},x} = 0$, so the momentum principle becomes

$$0 = \Delta p_{\text{sys},x} = \Delta p_{c,x} + \Delta p_{w,x} \tag{1}$$

From our diagram showing the change in the $x$-component of the momentum of the elements of the system, Eq. (1) becomes

$$0 = m_c \Delta v_c + \Delta m_w v_c(t) \tag{2}$$

where we can ignore the contribution form the second order term $\Delta m \Delta v_c$. Because the cart’s mass is increasing due to the material entering we have that

$$\Delta m_w = \Delta m_c \tag{3}$$

and so Eq. (2) can now be written after taking limits as $\Delta t \to 0$

$$m_c(t) \frac{dv_c}{dt} = - \frac{dm_c}{dt} v_c(t) \tag{4}$$

We can multiply both sides of Eq. (4) by $dt$ yielding

$$m_c(t) dv_c = -dm_c v_c(t) \tag{5}$$

This is now separable and collecting terms we have
We can integrate both sides paying careful attention to make a consistent choice for the limits of the integrals

\[ \int_{v_c'(0)}^{v_c'(t)} \frac{dv_c'}{v_c'} = - \int_{m_c'(0)}^{m_c'(t)} \frac{dm_c'}{m_c'}. \]

Integrating yields

\[ \ln \left( \frac{v_c(t)}{v_c(0)} \right) = - \ln \left( \frac{m_c(t)}{m_c(0)} \right) = \ln \left( \frac{m_c(0)}{m_c(t)} \right). \]

Exponentiating yields

\[ \frac{v_c(t)}{v_c(0)} = \frac{m_c(0)}{m_c(t)}. \]

The \( x \)-component of the velocity of the cart at time \( t \) is then

\[ v_c(t) = \frac{m_c(0)}{m_c(t)} v_c(0). \]

**Very Simple Alternative Method:**

Because the rain does not add any horizontal momentum and there are no external forces, the \( x \)-component of the momentum of the system must be constant. Therefore

\[ m_c(t)v_c(t) = m_c(0)v_c(0). \]

We immediately solve this equation for \( x \)-component of the velocity of the cart at time \( t \)

\[ v_c(t) = \frac{m_c(0)}{m_c(t)} v_c(0). \]

which is in agreement with our first approach.
Problem 8 Bicycling in the Rain

A bicycle rider is caught in a fierce rainstorm that begins at \( t = 0 \) when the rider is traveling at speed \( v_0 \). At ground level the rain is moving horizontally at the rider with speed \( u \) relative to the ground. The rider stops pedaling as the rain sticks to the rider. The initial mass of the bicyclist is \( m_0 \) (including mass of bicycle). You may ignore all resistance.

a) Derive a relation between the differential of the speed of the bicyclist, \( dv \), and the differential of the total mass of the bicyclist, \( dm \).

b) Integrate the above relation to find the speed of the bicyclist as a function of mass, \( v(m) \).

c) Assume that the rate that the mass is added to bicyclist is constant, \( \frac{dm}{dt} = b > 0 \), (this is a simplifying assumption that is only an approximation). How long does it take the bicyclist to come to a stop?

Solution:

Define the mass in our system as the mass \( m_b(t) \) of the bicyclist at time \( t \) and small mass of the rain that strikes and sticks to the bicyclist during the time interval \([t, t + \Delta t]\). We show the momentum diagrams for time \( t \) and \( t + \Delta t \) below.

The change in momentum of the system consisting of the bicyclist and the added rain is shown in the figure below.
There are no external forces in the $x$-direction so the momentum principle becomes

$$0 = m_b(t) \frac{dv_b}{dt} + \frac{dm}{dt} (v_b(t) + u).$$  \hfill (13)

We know that

$$\frac{dm}{dt} = \frac{dm}{dt}.$$  \hfill (14)

So Eq. (13) becomes

$$m_b(t) \frac{dv_b}{dt} = - \frac{dm}{dt} (v_b(t) + u).$$  \hfill (15)

After some rearrangement we have that

$$\frac{dv_b}{(v_b(t) + u)} = - \frac{dm}{m_b(t)}.$$  \hfill (16)

We can integrate both sides

$$\int_{v_b(t)}^{v_b(t)} \frac{dv_b}{(v_b(t) + u)} = - \int_{m_b(t)}^{m_b(t)} \frac{dm}{m_b(t)}.$$  \hfill (17)

yielding

$$\ln \left( \frac{v_b(t) + u}{v_0 + u} \right) = \ln \left( \frac{m_0}{m_b(t)} \right).$$  \hfill (18)

Exponentiation yields

$$\frac{v_b(t) + u}{v_0 + u} = \frac{m_0}{m_b(t)}.$$  \hfill (19)

We can solve Eq. (19) for the $x$-component of the velocity of the bicyclist at time $t$

$$v_b(t) = \frac{m_0}{m_b(t)} (v_0 + u) - u = u \left( \frac{m_0}{m_b(t)} - 1 \right) + v_0 \frac{m_0}{m_b(t)}.$$  \hfill (20)
Note that in the limit that $t \to \infty$, $m_b(t) \to \infty$, and $v_b(t) \to -u$ which is what we expect. The bicyclist moves backwards with the rain.