Problem 1 Marble Run

a) Fastest Speed A marble starts from rest and slides down hill. Which path leads to the highest speed at the finish?

1) 1
2) 2
3) 3
4) all result in the same final speed

Answer 4. We are assuming that there is no friction on the paths so the mechanical energy is constant. The change in potential energy is the same for all three paths therefore the change in kinetic energy is also the same for all three paths.

b) Shortest Time A marble starts from rest and slides down hill. Which path results in the shortest time to the finish?

1) 1
2) 2
3) 3
4) all result in the same final speed

Answer 3. We are assuming that there is no friction on the paths so the mechanical energy is constant. Because the speed is greatest on path 3, the horizontal component of the velocity is greater on path 3 than at any point on path 2 or path 1, (included the starting and ending segments). Therefore it takes less time on path 3 to traverse the horizontal distance between the starting and ending points of each path.
Problem 2: Escape Velocity from Moon

Find the escape speed of a rocket from the moon. Ignore the rotational motion of the moon. The mass of the moon is \( m = 7.36 \times 10^{22} \text{ kg} \). The radius of the moon is \( R = 1.74 \times 10^6 \text{ m} \). The universal gravitation constant \( G = 6.7 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2} \).

Solution:

The “escape velocity” (really the escape speed) is the magnitude of the velocity for which the net mechanical energy, the sum of the kinetic energy and the gravitational potential energy, is zero, where the gravitational potential energy is defined so that the potential energy goes to zero in the limit of an infinite distance. This means that at any finite distance the potential energy is negative, and hence the kinetic energy is positive, and the object is still moving with nonzero speed.

We have then that

\[
\frac{1}{2} m v_{\text{esc}}^2 - G \frac{m m_{\text{moon}}}{R} = 0,
\]

which is readily solved for

\[
v_{\text{esc}} = \sqrt{\frac{2 G m_{\text{moon}}}{R}} = \sqrt{\frac{2 \left(6.674 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}\right) (7.36 \times 10^{22} \text{ kg})}{1.74 \times 10^6 \text{ m}}} = 2.38 \times 10^3 \text{ m} \cdot \text{s}^{-1}.
\]
Problem 3: Roller Coaster

Consider a roller coaster in which cars start from rest at a height \( h_0 \), and roll down into a valley whose shape is circular with radius \( R \), and then up a mountain whose top is also circular with radius \( R \), as shown in the figure. Assume the contact between the car and the roller coaster is frictionless. The gravitational constant is \( g \). Assume that the wheels of the car run inside a track which follows the path shown in the figure below, so the car is constrained to follow the track.

a) Find an expression the speed of the cars at the bottom of the valley.

b) If the net force on the passengers is equal to \( 8 mg \) at the bottom of the valley, find an expression for the radius \( R \) of the arc of a circle that fits the bottom of the valley.

c) The top of the next mountain is an arc of a circle of the same radius \( R \). If the normal force between the car and the track is zero at the top of the mountain, what is the height \( h_{\text{top}} \) of the mountain?

Solution:

(a) Find an expression the speed of the car at the bottom of the valley.

Since we assume there is no loss of mechanical energy, the change in mechanical energy is zero,

\[
0 = \Delta E_{\text{mech}} = \Delta U + \Delta K = (U_f - U_i) + (K_f - K_i)
\]  \hspace{1cm} (1)

Set the potential energy at the ground to be zero.

Then
\[ U_i = mgh_0, \quad K_i = 0 \]
\[ U_f = 0, \quad K_f = \frac{1}{2}mv_{\text{bottom}}^2 \]

With these values Eq. (1) becomes

\[ 0 = (0 - mgh_0) + \left(\frac{1}{2}mv_{\text{bottom}}^2 - 0\right) \]

Thus we can solve for the speed at the bottom

\[ v_{\text{bottom}} = \sqrt{2gh_0} \]

(b) If the net force on the passengers is equal to \(8 \text{ mg}\) at the bottom of the valley, find an expression for the radius \(R\) of the arc of a circle that fits the bottom of the valley.

The free body diagram is shown in the figure below.

```
```

The force equation in the inward direction is then

\[ N_{\text{bottom}} - mg = \frac{mv_{\text{bottom}}^2}{R} \]

If the net force on the passengers is equal to \(8 \text{ mg}\), then

\[ N_{\text{bottom}} - mg = 8mg \]

Thus Eq. (5) becomes

\[ 8mg = \frac{mv_{\text{bottom}}^2}{R} \]

From Eq. (4) we have that
\[
\frac{mv^2_{\text{bottom}}}{R} = \frac{2mgh_0}{R}
\]  \hspace{1cm} (8)

After Substituting Eq. (8) into Eq. (7), yields

\[
8mg = \frac{2mgh_0}{R}
\]  \hspace{1cm} (9)

We can now solve for the radius of the circular trajectory at the bottom

\[
R = \frac{h_0}{4}
\]  \hspace{1cm} (10)

(c) The top of the next mountain is an arc of a circle of the same radius \( R \). If the car just loses contact with the road at the top of the mountain, what is the height \( h_{\text{top}} \) of the mountain?

At the top of the mountain the free body force diagram is shown in the figure below

Then the force equation becomes

\[
mg - N_{\text{top}} = \frac{mv^2_{\text{top}}}{R}
\]  \hspace{1cm} (11)

If the car just loses contact at the top, then \( N_{\text{top}} = 0 \), and Eq. (11) becomes

\[
mg = \frac{mv^2_{\text{top}}}{R}
\]  \hspace{1cm} (12)

We can use as our initial state, the original height and the final state the top of the mountain for our energy equation. Then we have that

\[
U_i = mgh_0 \quad , \quad K_i = 0
\]

\[
U_f = mgh_{\text{top}} \quad , \quad K_f = (1/2)mv^2_{\text{top}}
\]  \hspace{1cm} (13)

Eq. (1) becomes
Thus we can rewrite Eq. (14) after dividing through by \( R \) as

\[
\frac{mv_{\text{top}}}{R} = \frac{2mg(h_0 - h_{\text{top}})}{R}
\]  

(15)

Substituting Eq. (15) into Eq. (12) yields

\[
mg = \frac{2mg(h_0 - h_{\text{top}})}{R}
\]  

(16)

We can solve Eq. (16) for the \( h_{\text{top}} \)

\[
h_{\text{top}} = h_0 - \frac{R}{2}
\]  

(17)

Now substitute Eq. (10) and find that

\[
h_{\text{top}} = h_0 - \frac{h_0}{8} = \frac{7h_0}{8}
\]  

(18)
Problem 4: Ball and String

A ball of negligible size and mass $m$ hangs from a string of length $l$. It is hit in such a way that it then travels in a vertical circle. The initial speed of the ball after being struck is $v_0$. The goal of the first part of this problem is to find the tension in the string when the ball is at the top of the circle. You may assume that there are no external forces doing work on the ball and string. Let $g$ denote the magnitude of the gravitational constant.

a) Outline a strategy and then a plan for solving this problem. State any concepts you plan to use, include sketches and diagrams as needed.

b) Find the tension in the string when the ball is at the top of the circle. Express your answer in terms of $m$, $g$, $l$, and $v_0$ as needed.

When the ball is exactly at the top of the circle, it detaches from the string and follows the trajectory shown on the figure above. When the ball returns to the level of the bottom of the circle, it is a distance $d$ from the bottom of the circle.

c) Find the distance $d$. Express your answer in terms of $m$, $g$, $l$, and $v_0$ as needed.

Part (a)
The first part of the question clearly just involves circular motion. If I draw a free-body diagram for the ball at the top of the circle, I have a formula that tells me the total force this ball should experience. However, this formula involves the velocity of the ball at the top of the circle, so I’ll need to find that.

That said, my strategy will be as follows:

1. I’ll find the velocity of the ball at the top of the circle using conservation of energy.
2. I’ll use this to find the total force the ball should be experiencing, using the equations of circular motion.
3. I’ll draw a free-body diagram for the ball there.
4. I’ll use that to find the tension, $T$.

**Part (b)**

**Step 1** The easiest way to find the speed of the ball at the top of the circle is using conservation of energy. Let $v_0$ be the original speed of the ball, and $v_t$ be the speed at the top of the circle. The ball ends up a height $2l$ above its original stating point. So the energy balance is as follows

<table>
<thead>
<tr>
<th></th>
<th>Initially</th>
<th>Top of the circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kinetic energy</td>
<td>$\frac{1}{2}mv_0^2$</td>
<td>$\frac{1}{2}mv_t^2$</td>
</tr>
<tr>
<td>Potential energy</td>
<td>0</td>
<td>$2mg l$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>$\frac{1}{2}mv_0^2$</td>
<td>$\frac{1}{2}mv_t^2 + 2mg l$</td>
</tr>
</tbody>
</table>

There are no other forces, and so the total energy must be the same originally and at the top of the circle. So

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_t^2 + 2mg l$$
$$mv_0^2 = mv_t^2 + 4mg l$$
$$mv_t^2 = mv_0^2 - 4mg l$$
$$v_t^2 = v_0^2 - 4gl$$
$$v_t = \sqrt{v_0^2 - 4gl}$$

**Step 2** We know, from the equations of circular motion, that

$$F = \frac{mv_t^2}{l}$$

Feeding in the $v_t$ we found above

$$F = \frac{m(v_0^2 - 4gl)}{l}$$

And this force must point *downwards*, towards the centre of the circle.

**Step 3** A free-body diagram for the particle at the top of the circle is

```
   T
 / \\
mg
```

**Step 3** And so
\[
F = T + mg \\
m\left( v_0^2 - 4gl \right) \frac{1}{l} = T + mg \\
T = \frac{m\left( v_0^2 - 4gl \right)}{l} - mg \\
T = \frac{mv_0^2}{l} - 4mg - mg \\
T = \frac{mv_0^2}{l} - 5mg
\]

**Part (c)**

This is simply a case of projectile motion in 2-dimensions, which we covered earlier this year. At the top of the circle, the velocity is purely horizontal, and so our strategy is simply

1. Find the time taken for the particle to drop the height \(2l\) vertically, given that it started with 0 vertical velocity.

2. Work out how far it would have travelled horizontally in that time.

**Step 1** Original velocity is 0, acceleration is \(g\) and distance travelled is \(2l\), and so the time taken is

\[
x = v_0 t + \frac{1}{2} at^2 \\
2l = \frac{1}{2} gt^2 \\
4l = gt^2 \\
t = 2\sqrt{\frac{l}{g}}
\]

**Step 2** The original horizontal velocity (from part A) was

\[
v_t = \sqrt{v_0^2 - 4gl}
\]

And so the total distance travelled is

\[
d = tv_t \\
d = 2\sqrt{l} \sqrt{\frac{v_0^2 - 4gl}{g}} \\
d = 2\sqrt{\frac{l(v_0^2 - 4gl)}{g}}
\]
Block A of mass $m_A$ is moving horizontally with speed $V_A$ along a frictionless surface. It collides elastically with block B of mass $m_B$ that is initially at rest. After the collision block B enters a rough surface at $x = 0$ with a coefficient of kinetic friction that increases linearly with distance $\mu_k(x) = bx$ for $0 \leq x \leq d$, where $b$ is a positive constant. At $x = d$ block B collides with an unstretched spring with spring constant $k$ on a frictionless surface. The downward gravitational acceleration has magnitude $g$.

What is the distance the spring is compressed when block B first comes to rest? Express your answer in terms of $V_A$, $m_A$, $m_B$, $b$, $d$, $g$, and $k$. 

\[
F_{\text{ext}} = 0 \implies P_{x, i} = P_{x, f} \\
(m_A) V_A = (m_A) V_{Axf} + (m_B) V_{Bxf} \\
\Rightarrow V_A = V_{Axf} + \frac{m_B}{m_A} V_{Bxf} \quad (1) \\
\text{relative velocity} : \quad V_A = - (V_{Axf} - V_{Bxf}) \quad (2) \\
\text{add (1) and (2)} \\
2V_A = \left(1 + \frac{m_B}{m_A}\right) V_{Bxf} \\
V_{Bxf} = \frac{2V_A}{1 + \frac{m_B}{m_A}} = \frac{2m_A V_A}{m_A + m_B} \quad (3)
\]
\[ \begin{aligned}
&\text{Initial} \xrightarrow{d} \text{Final} \\
&\Box \rightarrow u_{Bx1} \\
&W_{nc} = E_f - E_i \\
&- \int b_x m g dx = \frac{1}{2} k x_f^2 - \frac{1}{2} m_B v_{Bx1}^2 \\
&- b m_B g \frac{d^2}{2} = \frac{1}{2} k x_f^2 - \frac{1}{2} m_B v_{Bx1}^2 \\
\Rightarrow & \quad x_f = \sqrt{\frac{m_B}{k} \left( v_{Bx1}^2 - b g d^2 \right)} \quad \text{Use eq. (3)}
\end{aligned} \]
Problem 6: Objects on a Ring

Two objects slide without friction on a circular ring of radius $R$ oriented in a vertical plane. The heavier object (of mass $3m$) is attached to a spring with an unstretched length of zero (admittedly an unphysical assumption) and spring constant $k$. The fixed end of the spring is attached to a point a horizontal distance $2R$ from the center of the circle. The lighter object (of mass $m$) is initially at rest at the bottom of the ring. The heavier object is released from rest at the top of the ring, then collides with and sticks to the lighter object. Find the value for $m$ that will allow the combined object (of mass $4m$) to just reach the point A on the ring, but go no higher. Express your answer in terms of some or all of the quantities $k$, $R$ and the acceleration of gravity $g$.

Solution

Let $p_0$ be the momentum of the $3m$ object before the collision. Set $U_{\text{gravity}} = 0$ at the bottom of the ring. There are no non-conservative forces. The kinetic energy at the bottom immediately before the collision is $K_b = \frac{p_0^2}{2(3m)}$. Energy is constant hence

$$K_i + U_i = K_b + U_b$$

$$0 + ((3m)g(2R) + (k/2)(\sqrt{5}R)^2 = \frac{p_0^2}{2(3m)} + (k/2)(\sqrt{5}R)^2$$

$$6mgR = \frac{p_0^2}{6m}$$

$$p_0 = 6m\sqrt{gR}$$

Momentum is conserved in the collision, but energy is not. The momentum then goes to zero at A. So the kinetic energy immediately after the collision is $K_a = \frac{p_0^2}{2(4m)}$. After the collision, energy is constant so

$$K_a + U_a = K_i + U_i$$
\[
\frac{p_0^2}{2(4m)} + \frac{(k/2)(\sqrt{5}R)^2}{k/2(3R)^2} = 0 + ((4m)gR + (k/2)(3R)^2)
\]
\[
\frac{36m^2gR}{8m} + (5k/2)(R^2) = 4mgR + (9k/2)(R^2)
\]
\[
\frac{mgR}{2} = 2kR^2
\]
\[
m = \frac{4kR}{g}
\]
Problem 7 Circular Orbits

A projectile of mass \( m \) is fired vertically from the earth’s surface with an initial speed that is equal to the escape velocity. The radius of the earth is \( R_e \), the mass of the earth is \( M_e \), and the universal gravitational constant is \( G \). Express your answers to the questions below in terms of \( M_e, R_e, m, \) and \( G \) as needed.

a) What is the initial speed of the projectile when it is launched from the surface of the earth?

When the projectile is a distance \( 2R_e \) from the center of the earth, it collides with a satellite of mass \( m \) that is orbiting the earth in a circular orbit. After the collision the two objects stick together. Assume that the collision is instantaneous.

b) What is the speed of the projectile, just before the collision, when it is a distance \( 2R_e \) from the center of the earth?

c) What is the speed of the satellite, just before the collision, when it is in a circular orbit of radius \( 2R_e \)?

d) What is the speed of projectile and satellite immediately after the collision?
The escape speed of the projectile occurs when the energy of the projectile-earth system is zero,

So

\[
\frac{1}{2} m_p v_{p,esc}^2 - \frac{G m_p m_e}{R_e} = 0
\]

So the escape speed is

\[
v_{p,esc} = \sqrt{\frac{2 G m_e}{R_e}}
\]

when the satellite reaches a height \( r = 2 R_e \), the energy is

\[
\frac{1}{2} m_p v_p^2 - \frac{G m_p m_e}{2 R_e} = 0.
\]

So the speed is

\[
v_p = \sqrt{\frac{G m_e}{R_e}}
\]

c) Newton’s Second Law for the satellite becomes

\[
\frac{G m_s m_e}{(2 R_e)^2} = \frac{m_s v_s^2}{2 R_e}
\]

So the velocity of the satellite is

\[
v_s = \sqrt{\frac{G m_e}{2 R_e}}
\]

d) Momentum is constant during the collision so

**Choose horizontal and vertical directions and** \( \theta \) **to be the angle with respect to the horizontal that the combined objects emerge after the collision. If the masses are equal** \( m_p = m_s = m \). **Then the momentum equations become**
vertical: \[ mv_p = 2mv_f \sin \theta \]

horizontal: \[ mv_s = 2mv_f \cos \theta \]

Square and add these two equations yields

\[ m^2 (v_s^2 + v_p^2) = 4m^2 v_f^2 \]

So

\[ v_f = \frac{1}{2} (v_s^2 + v_p^2)^{1/2} . \]

Substituting for the two speeds yields

\[ v_f = \frac{1}{2} \left( \left( \frac{Gm_e}{2R_e} \right) + \left( \frac{Gm_e}{R_e} \right) \right)^{1/2} = \sqrt{\frac{3Gm_e}{8R_e}} \]
Problem 8 Blocks and springs

A block of mass $m_b$ sits at rest on a frictionless table; the block has a circular surface of radius $R$ as shown in the figure. A small cube of mass $m_c$ and speed $v_{c,0}$ is incident upon the block; the cube slides without friction on the table and slides without friction up the block. At the top of the block, the cube compresses a spring of spring constant $k$ until it momentarily comes to rest a height $R$ above the table. The cube then slides back down until it leaves the block.

![Diagram of blocks and springs](image)

a) How much did the spring compress?

b) What is the final speed of the block when the cube is no longer on it?

Solution: If we take as our system, the cube and block, energy and momentum are both constant since the contact surface is frictionless and normal forces do no work. When the cube has reached its maximum height, the cube and block are moving with the same final speed $v_1$.

Therefore the momentum of the system is

$$m_c v_{c,0} = (m_c + m_b) v_1$$

which we can solve for the speed of the cube and block when the cube is at rest relative to the block.

$$v_1 = \frac{m_c v_{c,0}}{m_c + m_b}.$$
The energy of the system is
\[
\frac{1}{2} m v_{c,0}^2 = \frac{1}{2} (m_c + m_b) v_1^2 + m_c g R + \frac{1}{2} k x_f^2.
\] (21)

Substituting Eq. (20) into the energy condition yields
\[
\frac{1}{2} m v_{c,0}^2 = \frac{1}{2} (m_c + m_b) \left( \frac{m_c v_{c,0}}{m_c + m_b} \right)^2 + m_c g h + \frac{1}{2} k x_f^2.
\] (22)

We can rewrite this in terms of the change in potential energy
\[
\Delta U = \frac{1}{2} k x_f^2 + m_c g R = \frac{1}{2} m v_{c,0}^2 - \frac{1}{2} (m_c + m_b) \left( \frac{m_c v_{c,0}}{m_c + m_b} \right)^2 = \frac{1}{2} m v_{c,0}^2 \left( 1 - \frac{m_c}{m_c + m_b} \right)
\]
\[
= \frac{1}{2} \frac{m_c m_b}{m_c + m_b} v_{c,0}^2 = \frac{1}{2} \mu v_{c,0}^2
\] (23)

where
\[
\mu \equiv \frac{m_c m_b}{m_c + m_b}
\] (24)
is the reduced mass. This should not be surprising. Because the collision is elastic,
\[
\Delta U = -\Delta K = -\left( \frac{1}{2} \mu v_{rel,f}^2 - \frac{1}{2} \mu v_{rel,0}^2 \right).
\] (25)

The final relative speed \(v_{rel,f} = 0\) and the initial relative speed is \(v_{rel,0} = v_{c,0}\), so Eq. (25) becomes
\[
\frac{1}{2} k x_f^2 + m_c g R = \frac{1}{2} \mu v_{c,0}^2 \] (26)
in agreement with Eq. (23). So the amount that the spring was compressed is
\[
x_f = \sqrt{\frac{\mu v_{c,0}^2}{2 m_c g h} - \frac{2 m_c g h}{k}}.
\] (27)

We show the momentum diagram when the cube comes back down the block along with the momentum diagram before the cube went up the block in the figure below.
We can again apply the condition that the momentum of the system is constant,

\[ m_c v_{c,0} = m_b v_{b,2} - m_c v_{c,2} \]  \hspace{1cm} (28)

which can be rewritten as

\[ m_c (v_{c,0} + v_{c,2}) = m_b v_{b,2} \cdot \]  \hspace{1cm} (29)

The condition that the energy of the system is constant becomes

\[ \frac{1}{2} m_c v_{c,0}^2 = \frac{1}{2} m_c v_{c,2}^2 + \frac{1}{2} m_b v_{b,2}^2 \cdot \]  \hspace{1cm} (30)

We can rewrite the energy equation as

\[ m_c (v_{c,0}^2 - v_{c,2}^2) = m_b v_{b,2}^2 \]  \hspace{1cm} (31)

which after factoring becomes

\[ m_c (v_{c,0} - v_{c,2}) (v_{c,0} + v_{c,2}) = m_b v_{b,2}^2 \]  \hspace{1cm} (32)

Dividing Eq. (32) by Eq. (29) yields

\[ (v_{c,0} - v_{c,2}) = v_{b,2} \]  \hspace{1cm} (33)

or

\[ v_{c,2} = v_{c,0} - v_{b,2} \cdot \]  \hspace{1cm} (34)

Note that Eq. (34) can be rewritten as \( v_{c,0} = v_{b,2} + v_{c,2} \) which is just our condition that the relative speed remains unchanged \( v_{rel,0} = v_{rel,2} \). We can substitute Eq. (34) into Eq. (28) and find that

\[ m_c v_{c,0} = m_b v_{b,2} - m_c (v_{c,0} - v_{b,2}) \cdot \]  \hspace{1cm} (35)
We can solve Eq. (35) for the speed of the block when the cube leaves the block,

\[ v_{b,2} = \frac{2m_c v_{c,0}}{m_b + m_c} \]  

Suppose we view the condition from the center of mass frame.

\[ v_{b,0}' = \frac{m_b v_{c,0}}{m_b + m_c} \]  

Then the block has an initial speed equal to the center of mass speed

\[ v_{b,2}' = v_{b,0}' = \frac{m_b v_{c,0}}{m_b + m_c} \]  

After the collision the block reverses direction but maintains the same speed in the center of mass frame so

\[ v_{b,2}' = v_{b,0}' = \frac{m_b v_{c,0}}{m_b + m_c} \]  

In the lab frame the speed of the block is then

\[ v_{b,2} = v_{b,2}' + v_c = \frac{m_c v_{c,0}}{m_b + m_c} + \frac{m_c v_{c,0}}{m_b + m_c} = \frac{2m_c v_{c,0}}{m_b + m_c} \]  

Agreeing with Eq. (36).